Holographic entanglement entropy: temperature and flavour contributions

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Abstract

This thesis investigates entanglement in strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory by applying the AdS/CFT correspondence to calculate the entanglement entropy holographically. I have derived an analytic expression for the entanglement entropy at finite temperature, which agrees with known numerical results. My result has a temperature-dependent area law, which indicates a thermally induced effective mass for the adjoint degrees of freedom. This phenomenon is known as Debye screening.

Furthermore, I have considered the leading order correction to the entanglement entropy caused by coupling the theory to massive flavour degrees of freedom. My results show a mass-dependent contribution to the area law, which indicates a modification of the Debye screening. Furthermore, my findings reproduce the known results for the thermal entropy, as well as the results for the thermal potentials deduced from it. These display the ‘meson melting’ transition between stable and unstable mesons.
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Chapter 1
Introduction

In this thesis, I examine the entanglement entropy in strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM theory). In quantum field theory, this measure for the quantum entanglement is difficult to calculate for higher-dimensional theories. This is true in particular at strong coupling, where perturbative calculations are no longer possible. A promising approach to this problem is to consider a dual theory, which describes the same physics albeit looking completely different. Various dualities, where two different theories are only different formulations of the same dynamics, emerge from physics and especially from string theory. One example is the AdS/CFT correspondence, which maps $\mathcal{N} = 4$ SYM theory to a five-dimensional theory containing gravity. Although gauge theory and gravity are completely different theories, the connection between them was already anticipated before the discovery of the AdS/CFT correspondence. Let us begin by looking at these arguments.

First clues

![Diagram](image.png)

Figure 1.1: Vacuum amplitudes and corresponding string theory diagrams

There were already clues about a link between gauge theory and string theory or gravity in the 70’s and 90’s. In 1974, ’t Hooft examined strongly coupled gauge theories with gauge group $SU(N)$ [1]. Since it is not possible to make an expansion in the coupling constant $g_{YM}$, he treated $N$ as a free parameter and expanded the theory for large $N$, while keeping the effective coupling constant $g_{YM}^2 \cdot N$ fixed. Interestingly, Feynman diagrams with scaling $1/N^{2-2g}$ can be mapped to string theory diagrams with $g$ handles (c.f. Figure 1.1). The expansion in $1/N$ is therefore equivalent to an
perturbative expansion of string theory with string coupling $g_s \sim 1/N$. However, the precise form of the dual string theory is unknown and is difficult to construct for higher dimensions.

The second clue was the Bekenstein entropy bound [3] and the thereby motivated holographic principle [4–6]. The Bekenstein bound states that the entropy in a volume cannot exceed $A/4G_N$, where $A$ is the boundary area and $G_N$ is the Newton’s constant. The holographic principle states that for a theory of quantum gravity, the degrees of freedom in a volume can be described by a theory living on the boundary, in which the degrees of freedom scale with the area. Therefore, quantum gravity in $D$ dimensions can be described by some theory without gravity in $D-1$ dimensions.

AdS/CFT correspondence

In the 1960s, string theory\(^1\) was developed to describe strong interactions. Although it was possible to describe confinement, it was not possible to describe the high energy behaviour of hadron physics and this approach was abandoned. However, string theory is a promising candidate for a unified theory describing weak force, strong force, electromagnetic force and gravity. The fundamental objects of string theory are open and closed strings. All particles are excitations of this fundamental objects. The theory automatically contains gauge symmetry as well as gravity, because gravitons are excitations of closed strings and gauge fields are excitations of open strings. However, consistency of string theory requires 10 spacetime dimensions, making it a non-trivial problem to obtain a four-dimensional theory.

In the 1970s, quantum chromodynamics (QCD) was developed to describe the strong interaction. It is $SU(3)$ gauge theory, which is asymptotically free (i.e. has small coupling) at high energies and confined (i.e. has large coupling and only uncharged bound states exist) at low energies. As a consequence, it is not possible to perform perturbative calculations at low energy and complex numerical calculations are necessary.

In 1997, Maldacena proposed the AdS/CFT correspondence\(^2\), which is motivated by dualities between different perspectives in string theory. The correspondence maps higher dimensional theories in a weakly curved spacetime to strongly coupled, large $N_c$ conformal field theories (CFTs) with gauge group $SU(N_c)$. The spacetime on the gravity side is Anti-de Sitter (AdS) times a compact space. Symmetry considerations place the field theory on the boundary of the AdS space, making the AdS/CFT duality a concrete realisation of the holographic principle. The additional radial coordinate in AdS is identified with the energy scale in the field theory, where the boundary corresponds to high energy and the deep bulk to low energy. Both sides of the correspondence describe the same dynamics, the same symmetries and there is a one-to-one map between operators and fields [12, 13]. The power of the correspondence is that it

\(^1\)An extensive explanation of string theory is e.g. given in [7], short introductions can be found in [2,8,9].

\(^2\)The original proposal was made in [10], reviews can be found in [2,8,9,11].
One particularly interesting example is the duality between a field theory in four dimensions and a theory containing gravity in five dimensions. The field theory is \( \mathcal{N} = 4 \) SU\((N)\) supersymmetric Yang-Mills theory (or short \( \mathcal{N} = 4 \) SYM theory), which is dual to a theory in AdS\(_5 \times S^5\). This duality already passed a number of non-trivial tests (e.g. the calculation of the correct three point functions and of the conformal anomaly).

Even if a dual description for QCD is not known, the AdS/CFT correspondence can provide insights into the strong coupling behaviour. The hope is that by studying supersymmetric theories, we obtain an intuition for the behaviour of strongly coupled theories and that some observables have an universal behaviour. This universal behaviour was already observed for the ratio between shear viscosity \( \eta \) and entropy \( s \). The AdS/CFT correspondence yields the famous result \( \eta/s = 1/4\pi \) [14] for a large class of strongly coupled theories, which is a small value and of the same order of magnitude as experimental results for QCD at high temperature. Furthermore, QCD may be similar to supersymmetric gauge theories in a specific temperature range. Above the deconfinement temperature \( T_c \), the theory is still strongly coupled and shares properties (e.g. Debye screening) with \( \mathcal{N} = 4 \) SYM theory, as can be seen in Figure 1.2. Debye screening is the phenomenon that a massless theory develops a finite correlation length due to a thermally induced mass.
Entanglement entropy

Besides understanding local quantities like correlation functions, the AdS/CFT correspondence can also be used to study non-local properties like the entanglement entropy\(^{3}\). This entropy is a measure for how entangled or strongly correlated a spatial region \(B\) is with its complement \(C\). For a global state \(\rho\), it is defined as the von Neumann entropy of the reduced density matrix

\[
\rho_B = \text{Tr}_C \rho, \\
S_{EE} = - \text{Tr}_B \rho_B \ln \rho_B.
\]

The entanglement entropy is the missing information for an observer, who has only access to the region \(B\) but not to its complement. The physical importance of the entanglement entropy is that it is a measure for the effective degrees of freedom and can be used as an order parameter for quantum phase transitions. It is not always possible to describe such phase transitions by classical order parameter.

Although the equation for the field theory side looks simple, it involves complicated quantum calculations. There are results for two-dimensional conformal field theories, but not many for higher-dimensional theories. This makes it particularly interesting to consider a dual description. Inspired by the holographic principle and the entropy bound, Ryu and Takayanagi conjectured that the smearing out of the complement corresponds to hiding part of the bulk from the observer. The hidden information is bounded by the Bekenstein entropy bound. By minimizing the surface area of the hidden bulk volume, a bound for the corresponding entropy is found.\(^{4}\) The entanglement entropy of a region is therefore bounded by \(A/4G_N\), where \(A\) is the area of the minimal surface, as shown in Figure 1.3. Calculations in two dimensions showed that this bound is saturated, suggesting that this is also the case in higher dimensions. Therefore, a

\(^{3}\)For information about entanglement on the field theory side see [15–17], for the holographic dual see [18,19] or the reviews [20,21].

\(^{4}\)This minimizing is performed on a constant time slice. This only works for static spacetimes, where a timelike killing field defines a unique foliation. A covariant holographic entropy was proposed in [22].
complicated quantum calculation reduces to a geometrical calculation. This gives a geometrical meaning to the entanglement entropy in the field theory.

**Black holes**

One important generalization of the AdS/CFT correspondence is the generalization to non-vanishing temperature by placing a black hole in the AdS space [23]. This connects the AdS/CFT duality to open questions in black hole physics. Black holes have laws analogous to the laws of thermodynamics. In particular, it has a temperature, whose black body radiation manifests itself as Hawking radiation [24,25], and an entropy [26]. The entropy of a black hole is the Bekenstein entropy

\[ S_{BH} = \frac{A}{4G_N}, \]

where \( A \) is the horizon area. For a statistical system, the entropy scales in general with the volume, hence the black hole entropy behaves like the entropy of a statistical system living in one spacetime dimension less. Furthermore, the exact origin of this entropy is an open question. In statistical mechanics, the entropy is a measure for possible microscopic states described by the same macroscopic variables. For black holes, counting the microscopic states would require a complete understanding of quantum gravity, which is not yet solved. However, semi-classical calculations of the entropy by using the saddle point approximation of the Euclidean path integral confirm the Bekenstein entropy. The AdS/CFT correspondence offers a new perspective of this entropy: the black hole entropy in AdS can be interpreted as the statistical entropy of a gauge theory living on the boundary of this space. Furthermore, since the field theory side of the correspondence is unitary, the gravity side also suspected to be unitary\(^5\). However, it is still an open question how unitarity is restored.

**Flavour in AdS/CFT**

Maldacena’s original proposal only describes adjoint degrees of freedom on the field theory side. However, it is well known that QCD also contains flavours, which are fundamental degrees of freedom. Introducing \( N_f \) flavour on the field theory side is dual to placing a stack of \( N_f \) hyperplanes in the gravity side [9,28]. Open strings beginning at the hyperplane and falling into the bulk are transforming in the fundamental representation of \( SU(N_c) \) and yield the fundamental degrees of freedom.

In general, the hyperplanes deform the space and it is necessary to solve the equation of motion again instead of working with the AdS background. However, some behaviour of the theory can be examined in the probe approximation, which means the deformation of the background is neglected. One example for such a property is the behaviour for massive flavour. Massive flavour are introduced by embedding the

\(^5\)For a discussion, see e.g. [27].
hyperplanes non-trivially in the compact space. For zero temperature, the size of the hyperplanes shrinks to zero at some radial position and the hyperplanes end. Since the radial direction corresponds to an energy scale of the field theory, this represents a mass gap in the dual description. The quark condensate can be read of from the embedding and is vanishing at zero temperature.

Going to non-vanishing temperature, the situation changes. The hyperplanes only fill the space outside the event horizon and there are two different kinds of embeddings: embeddings with large mass which end before the horizon (Minkowski embeddings) and embeddings with small mass which reach the horizon (black hole embeddings). Since the only energy scales are temperature and mass, only the ratio between them is important, making large (small) mass equivalent to low (high) temperature. It has been shown that mesons (i.e. the supersymmetric versions of a quark anti-quark bound states) are stable at low temperature, but have a finite decay time at high temperature. The transition between these two phases is not smooth, but a first order phase transition because of a discontinuous jump between the two types of embeddings [29–32]. This phase transition is called "meson melting". Additionally, for non-vanishing mass and non-vanishing temperature, there is a quark condensate.\footnote{Chiral symmetry breaking, i.e. a quark condensate at vanishing mass, is prohibited by supersymmetry.}

However, other properties cannot be examined in this approximation, e.g. the entanglement entropy. Therefore, it is necessary to leave the probe approximation and treat the flavour as small perturbation of the background. This can be solved order by order. For the entanglement entropy, a method to calculate the leading order flavour correction was worked out in [33]. Because of the extremity of the minimizing surface, it is only necessary to calculate the area change for the same surface. This was already applied to massive flavour at zero temperature [34, 35] and massless flavour at finite temperature and charge density [36].

**Results**

This thesis examines the entanglement in strongly coupled $\mathcal{N} = 4$ SYM theory by applying Maldacena’s AdS/CFT correspondence and Ryu’s and Takayanagi’s proposal for the holographic entanglement entropy. My results are split in two parts.

The first part analyses the entanglement entropy of the adjoint degrees of freedom in $\mathcal{N} = 4$ SYM theory at finite temperature. In [18,19], the holographic entanglement entropy for a two-dimensional conformal field theory dual to AdS$_3$ was calculated, while limiting the discussion of higher-dimensional cases to general arguments about the qualitative behaviour. Up to now, there are only numerical results for the holographic entanglement entropy at finite temperature in higher-dimensions (see [37]). I have obtained an analytical expressions for this entanglement entropy for a straight belt (c.f. Figure 1.3). The entanglement entropy and the width of the straight belt are expressed in terms of the turning point of the minimal surface. My results agree with the known fact that in the large-volume limit, the entanglement entropy agrees
with the thermal entropy. After subtracting the thermal contribution, the ‘corrected’
entanglement entropy is lower than the zero temperature result and follows an area
law in the large-volume limit, which is due to the Debye screening of the plasma.
Furthermore, I have analysed the effective degrees of freedom by a proposal for a
generalized entropic c-function in [20]. The effective degrees of freedom agree with
the zero temperature theory in the UV, decrease along the RG flow and vanish at the
IR fixed point. This agrees with the physical interpretation that the adjoint degrees
of freedom have a thermally induced mass and are integrated out in the low energy
theory.

Furthermore, I calculate the leading order flavour correction to the entanglement
entropy for massive flavour, following [33] and generalizing the calculations performed
in [34–36]. For the backreaction of the metric, the initial conditions are chosen such that
the contribution from the thermal entropy separates. The thermal entropy obtained
in this thesis is used to determine the free energy and average energy, reproducing
the results in [38] and showing the ‘meson melting’ phase transition. Furthermore, I
examine the correction of the ‘corrected’ entanglement entropy (i.e. the entanglement
entropy without thermal contribution) and compare the results with the zero temper-
ature results. For vanishing mass, the flavour correction to the entanglement entropy
is proportional to the entanglement entropy of the adjoint degrees of freedom, which
was already observed for vanishing temperature. This implies that the entanglement
entropy in a theory with flavour is equivalent to the entanglement entropy in a theory
with increased number of adjoint degrees of freedom. For finite mass, the entanglement
for a small volume is equivalent to the zero temperature result. In the large-volume
limit, the entanglement approaches a constant, which is in general different from the
constant of the zero temperature result. This asymptotic constant agrees with the zero
temperature result for heavy flavours. Moreover, it is larger than the zero temperature
result near the phase transition and lower for high temperatures. This indicates that
there is a range where the effective flavour mass is lower than the effective mass of the
adjoint degrees of freedom, which decreases the effects of the Debye screening. For high
temperature, i.e. in the phase where mesons are melted, the temperature may induce
a mass increase for the flavour degrees of freedom, which would enhance the effects of
the Debye screening.

Outline

This thesis is structured as follows. The first chapter is a review of the basics of
AdS/CFT. At the beginning, this chapter focuses on both sides of the duality, i.e. $\mathcal{N} = 4$ SYM theory at the field theory side with additional focus on thermodynamics and the
entanglement entropy as well as AdS spacetimes at the gravity side. An additionally
important point in understanding the AdS/CFT correspondence and specifically its
origin is string theory. Therefore, we review the key ideas with special emphasis on
supergravity and D-branes. Subsequently, we state the precise form of the Maldacena’s
AdS/CFT correspondence and look at the mapping between gravity theory and field
theory. A main emphasis is on adding flavour degrees of freedom and the holographic dual of the entanglement entropy.

As explained above, the results of this thesis are split into two parts. The analysis of the entanglement entropy in $\mathcal{N} = 4$ SYM theory is presented in the second chapter. The leading order flavour correction to the entanglement entropy is calculated in chapter 3. In chapter 4, I summarize my results and give an outlook on open questions. The appendix of this thesis contains a summary of conventions used (in particular of the different ranges of indices) and an overview of the AdS/CFT dictionary, which maps quantities from the gravity side to quantities of the field theory side. Furthermore, this part contains a short review of hypergeometric functions and some technical analytical calculations. Lastly, I summarize my numerical calculations.
Chapter 2

Preliminaries: AdS/CFT

In this chapter, the knowledge important for understanding my calculations is reviewed. The starting point are the two theories the AdS/CFT correspondence relates: conformal field theory (CFT) and gravity in Anti-de Sitter space (AdS). After briefly reviewing the basics of string theory, the origin and the exact form of the duality are summarized. Since the original duality only contains adjoint degrees of freedom, we place special emphasis on the necessary modifications to obtain flavour degrees of freedom. Furthermore, the entanglement entropy as measure for entanglement is introduced.

During preparation of this thesis, some up-to-date books about the AdS/CFT correspondence appeared, which are [2,8,39]. More information about string theory can be found in [7]. The following sections will follow these books as well as further references, as mentioned in the text.

2.1 Field theory

2.1.1 $\mathcal{N} = 4$ SYM theory

$\mathcal{N} = 4$ SU($N$) supersymmetric Yang-Mills theory (or short $\mathcal{N} = 4$ SYM theory) is a conformal field theory with R-symmetry group $SU(4)_R$. In the large $N$ limit, it is exactly solvable and can be used as a simple toy model for interacting field theories in four dimensions. Let us shortly examine the different kind of symmetries in this theory.

Conformal symmetry

The conformal symmetry is a special case of Weyl symmetry. A Weyl transformation rescales the metric by multiplying a positive function

$$g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g_{\mu\nu}(x). \quad (2.1)$$

This transformation preserves angles and causality. Conformal transformations are the restriction of Weyl transformations to flat spacetime (i.e. $g_{\mu\nu} = \eta_{\mu\nu}$). An infinitesimal
### Table 2.1: Conformal transformations

<table>
<thead>
<tr>
<th>$\epsilon^\mu(x)$</th>
<th>Dimension</th>
<th>Transformation</th>
<th>Generator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^\mu$</td>
<td>$d$</td>
<td>Translation</td>
<td>$P^\mu$</td>
</tr>
<tr>
<td>$+ \quad w^{[\mu\nu]} x_\nu$</td>
<td>$d(d-1)/2$</td>
<td>Lorentz transformation</td>
<td>$J^{\mu\nu}$</td>
</tr>
<tr>
<td>$+ \quad \lambda x^\mu$</td>
<td>$1$</td>
<td>Dilatation</td>
<td>$D$</td>
</tr>
<tr>
<td>$+ \quad b^\mu x^2 - 2 b^\nu x_\nu x^\mu$</td>
<td>$d$</td>
<td>Special conformal transformation</td>
<td>$K^\mu$</td>
</tr>
</tbody>
</table>

Conformal transformation can be achieved by the coordinate transformation

$$x^\mu \rightarrow x^\mu + \epsilon^\mu(x),$$

where $\epsilon$ satisfies the conformal Killing equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\sigma(x) \eta_{\mu\nu}. \quad (2.3)$$

For spacetime dimension $d > 2$, the conformal group extends the Poincaré group with dilatations and special conformal transformations, as shown in Table 2.1. This transformation rescales the metric by multiplying $\exp(2\sigma(x))$, where $\sigma = \lambda - 2b^\mu x_\mu$.

The conformal symmetry group is equivalent to $SO(d,2)$.

A theory which is invariant under dilatations is called scale invariant and contains only dimensionless parameters. Physically, such a theory is independent of the energy scale of interest. An infinitesimal dilatation causes $\delta\lambda g_{\mu\nu} = 2\lambda \delta g_{\mu\nu}$. In a classical scale invariant theory, the variation of the action vanishes, which leads to a traceless stress energy tensor.

$$\delta\lambda S = -\frac{\lambda\sqrt{-g}}{2} T^{\mu\nu} g_{\mu\nu} = 0 \quad (2.4)$$

When considering a quantum theory, the expectation value of the trace can be non-vanishing due to the conformal anomaly, also called Weyl anomaly. For $d = 4$, the expectation value of the trace is

$$\langle T^{\mu\mu} \rangle = \frac{c}{16\pi^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E, \quad (2.5a)$$

where $E$ is the Euler topological density and $C$ is the Weyl tensor.

---

1. $d = 2$ is special, because the conformal killing equation reduces to the Cauchy-Riemann differential equation and $e^0 + ie^1$ is an analytic (i.e. holomorphic) function of $x^0 + x^1$. The symmetry group in $d = 2$ is infinite dimensional.

2. This is the case in Minkowski signature. In Euclidean signature, it is equivalent to $SO(d+1,1)$.

3. The Euler topological density and the Weyl tensor are

$$E = R^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} - 4 R^{\mu\nu} R_{\mu\nu} + R^2,$$

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{d-2} (g_{\mu\rho} R_{\nu\sigma} - g_{\nu\rho} R_{\mu\sigma}) + \frac{2}{(d-1)(d-2)} R g_{\mu\rho} g_{\nu\sigma}. $$
### 2.1 Field theory

#### Table 2.2: Massless supersymmetric multiplets

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$N = 1$</th>
<th>$N = 1$</th>
<th>$N = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>1/2</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>0</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>-1/2</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>-1</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
</tbody>
</table>

In general, the conformal symmetry can be broken in the renormalized theory (as happens e.g. in quantum electrodynamics).

**Supersymmetry**

The Coleman-Mandula theorem restricts the bosonic symmetries of a theory to Poincaré symmetry and internal symmetries. To bypass this, we have to use supersymmetry with spinor charges $Q^a$, $a \in 1, \ldots, N$. In $d = 4$, this charges can be written as left-handed Weyl spinor $Q^a_\alpha$ and right-handed Weyl spinor $\bar{Q}^a_{\dot{\alpha}} = (Q^a_\alpha)^*$, where $\alpha, \dot{\alpha} \in \{1, 2\}$ are spinor indices.\(^4\) When extending the Poincaré symmetry, the mass is still a Casimir operator, but the helicity is not. $Q^a_\alpha$ decreases the helicity by $1/2$, whereas $\bar{Q}^a_{\dot{\alpha}}$ increases it by $1/2$. An irreducible representation of the supersymmetry algebra has the following properties:

- same mass $m$ for all states,
- contains states with different helicity,
- the number of fermionic and bosonic degrees of freedom agree,
- all states are in same representation of the gauge group.

The size of multiplets depends on whether they are massive or not. In a massless representation, $Q^a_2$ acts trivially on states, i.e. $Q^a_2|\text{phys}\rangle = 0$. Starting with a state of highest helicity, all $2^N$ states of the multiplet can be constructed by acting with $Q^a_1$. For CPT-invariance, we also have to add the states with negative helicity. The following interesting multiplets are shown in Table 2.2. The names of the multiplets depend on the highest helicity state. A gauge multiplet has $\lambda_{\text{max}} = 1$ and a chiral multiplet has $\lambda_{\text{max}} = 1/2$.

The particle content of $\mathcal{N} = 4$ SYM theory is that of the $\mathcal{N} = 4$ gauge multiplet. It has the same particle content as a $\mathcal{N} = 1$ gauge multiplet and three $\mathcal{N} = 1$ chiral multiplets. It is easier to obtain the unique supersymmetric theory by considering these $\mathcal{N} = 1$ multiplets.

\(^4\)Weyl spinors are possible in even dimensions. The dimension of a spinor is dimension-dependent, e.g. 4 for $d = 4$ and 16 for $d = 10$. 

For massive multiplets, $Q^a_\alpha$, $\alpha \in 1, 2$ act in a non-trivial way in general. For a special choice of algebra however, up to $\mathcal{N}/2$ linear combinations (with respect to the index $a = 1, \ldots, \mathcal{N}$) of them act trivially for both spinor indices. Therefore, the number of creation operator is reduced to 2 ($\mathcal{N} - k$), yielding $2^{2(\mathcal{N} - k)}$ states in the massive multiplets. This multiplets are called 1/2 Bogomolnyi-Prasad-Sommerfield multiplets or short BPS multiplets. For this states, $k$ of the central charges have the maximal value for a given mass $m$ or equivalently, this states are the lightest particles for given central charges. This ensures stability of the theory.

$\mathcal{N} = 4$ SU$(\mathcal{N})$ supersymmetric Yang-Mills theory

The particle content of the massless $\mathcal{N} = 4$ supersymmetric multiplet consists of a gauge field $A_\mu$, four Weyl fermions $\lambda^a$ and six real scalars $\phi^i$. All of this fields transform in the adjoint representation of SU$(\mathcal{N})$. The only CP-invariant Lagrangian with the required symmetries is

$$
\mathcal{L} = \text{Tr} \left[ -\frac{1}{2g_{YM}^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i \lambda^a \tilde{\sigma}^\mu D_\mu \lambda_a - D_\mu X^i D^\mu X^i 
+ \frac{g_{YM}}{2} C_{iab} \lambda_a [X^i, \lambda_b] + C_{iab} \lambda^a [X^i, \lambda^b]
+ \frac{g_{YM}^2}{2} [X^i, X^j]^2 \right],
$$

(2.6)

where $g_{YM}$ is the Yang-Mills coupling, $F_{\mu\nu}$ is the field strength tensor of $A_\mu$ and $C_{iab}$ are Clebsch-Gordan coefficients.\footnote{Used conventions: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu]$, $D_\mu = \partial_\mu + i [A_\mu, \cdot]$, $\tilde{\sigma}^\mu = (-1_2, -\sigma^I)$.} Since the only parameter in the Lagrangian is the dimensionless coupling constant $g_{YM}$, the theory is conformal on the classical level. A special feature of $\mathcal{N} = 4$ SYM theory is that it is also scale invariant as a quantum theory.

Due to the conformal symmetry, the number of supercharges is doubled. The (Poincaré) supercharges $Q^a$, $a = 1, \ldots, \mathcal{N}$ are the superpartners of $P^\mu$. In a theory with conformal symmetry, we have to introduce additional (special conformal) supercharges $S^a$ as superpartners of $K^\mu$. Since a spinor in four dimensions has dimension 4, this results in 32 supercharges.

2.1.2 Entanglement entropy

In this thesis, I calculate the entanglement entropy in $\mathcal{N} = 4$ SYM theory. It is a measure for the entanglement between a spacelike submanifold $B$ and its complement $C$. A common choice for $B$ is a straight belt with width $l$ or a circular disk with radius $r$, as shown in Figure 2.1. Physically, the entanglement entropy can be understood as the missing information for an observer who only has access to the region $B$ but not to its complement. In quantum field theory, the entanglement entropy is a measure for the degrees of freedom of the theory and can be used as an order parameter for quantum phase transitions. The following review is based and the introductory papers by Calabrese [15–17], Takayanagi’s review in [19] and [40].
2.1 Field theory

\[ \rho_B = \text{Tr}_C \rho. \quad (2.7) \]

The entanglement entropy \( S_{EE} \) is defined as the von Neumann entropy of the reduced density matrix

\[ S_{EE} = -\text{Tr}_B \rho_B \ln \rho_B. \quad (2.8) \]

Let us have a look on how this entanglement entropy works as a measure for entanglement. We consider a pure state

\[ \rho = |\psi\rangle \langle \psi|, \quad (2.9) \]

where the Hilbert space \( \mathcal{H} \) can be written as a product space \( \mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_C \). The state \( |\psi\rangle \) can be written in the Schmidt decomposition

\[ |\psi\rangle = \sum_i a_i |\psi_i\rangle_B \otimes |\psi_i\rangle_C, \quad \text{with } \sum_i |a_i|^2 = 1, \quad (2.10) \]

where \( \{ |\psi_i\rangle_B \} \) and \( \{ |\psi_i\rangle_C \} \) form an orthonormal basis of \( \mathcal{H}_B \) and \( \mathcal{H}_C \) respectively. The reduced density matrix is

\[ \rho_B = \text{Tr}_B \rho, \]

\[ = \sum_i |a_i|^2 \cdot |\psi_i\rangle_B \langle \psi_i|_B. \quad (2.11) \]

We see that although \( \rho \) is pure (i.e. \( \sum_i |\lambda_i|^2 = 1 \)), the reduced density matrices \( \rho_B \) and \( \rho_C \) are in general not because

\[ \text{Tr}_B \left( \rho_B^2 \right) = \text{Tr}_C \left( \rho_C^2 \right) = \sum_i |a_i|^4. \quad (2.12) \]
The entanglement entropy is

\[ S_{EE}(B) = -\sum_i |a_i^2| \ln |a_i^2| = S_{EE}(C) \]  

(2.13)

and in general non-vanishing. It only vanishes for product states, i.e. for

\[ |\psi\rangle = |\psi\rangle_B \otimes |\psi\rangle_C, \]  

(2.14)

which means that the two subsystems are not entangled. This makes the entanglement entropy to a measure for the entanglement.

Calculating the full reduced density matrix or even only its eigenvalues \( \lambda_i \) is difficult. Due to the properties of density matrices, the eigenvalues are all positive, in the interval \([0, 1]\) and sum up to 1. Therefore, \( \text{Tr}_B \rho_B^n = \sum_i \lambda_i^n \) is absolutely convergent for \( n \geq 1 \) and can be analytically continued to \( \text{Re}(n) \geq 1 \). This leads to the definition of the Rényi entropies \( S^{(n)} \)

\[ S^{(n)} = \frac{1}{1-n} \ln \text{Tr}_B \rho_B^n. \]  

(2.15)

In the limit \( n \to 1^+ \), this expression approaches the entanglement entropy

\[ \lim_{n \to 1} S^{(n)} = \lim_{n \to 1} \frac{-1}{n-1} \ln \left( \text{Tr}_B \rho_B^n \right), \]

\[ = -\partial_n \text{Tr}_B \rho_B^n|_{n=1}, \]

\[ = S_{EE}. \]  

(2.16)

\( \text{Tr}_B \rho_B^n \) is the partition function on an n-sheeted Riemann surface, where the region \( B \) at a fixed time is removed and the boundaries are sewed together. This works well in two dimensions, but there are no exact results for higher dimensions.

This makes it difficult to prove whether a dual quantity yields the same result. Therefore, let us take a look at some qualitative features of the entanglement entropy, which can be checked for a candidate of a dual. One well understood contribution comes from short-range correlation. At the boundary of the region, shorter and shorter correlation become important. To regulate this contribution, we introduce a UV cut-off proportional to \( 1/\epsilon \). In \( d \) spacetime dimensions, this short-range contribution yields an area law \([41–43]\) in leading order

\[ S_{EE} \propto \text{Vol}(\partial B) \begin{cases} \frac{1}{\epsilon^2} & d > 2 \\ \log \epsilon & d = 2 \end{cases}, \]  

(2.17)

which is proportional to the boundary of \( B \). Together with subleading contributions, the entanglement entropy can be written as an expansion in \( \epsilon \) (c.f. \([44]\))

\[ S_{EE} = c_{d-2}(\partial B)\epsilon^{-(d-2)} + c_{d-1}(\partial B)\epsilon^{-(d-1)} + \cdots + c_0(\partial B) \ln(\epsilon) + \text{finite}, \]  

(2.18)

where only the finite part depends on the volume.

In a theory with a mass gap, the momentum of the short-range excitations is also limited from below. Therefore, we expect an additional area contribution. Furthermore, the entanglement entropy has following properties
• for a pure state: $S_{EE}(B) = S_{EE}(C)$,

• subadditivity: for a region $B$ which is divided into $B_1$ and $B_2$

$$S_{EE}(B_1) + S_{EE}(B_2) \geq S_{EE}(B),$$ (2.19)

• strong subadditivity: for non-intersecting regions $B_i$

$$S_{EE}(B_1 + B_2 + B_3) + S_{EE}(B_2) \leq S_{EE}(B_1 + B_2) + S_{EE}(B_2 + B_3).$$ (2.20)

To obtain some intuition what happens to the entanglement entropy in different situations, let us review results for two dimensions, where $B$ is an interval with width $l$. For a conformal field theory with central charge $c$, the entanglement entropy is

$$S_{EE} = \frac{2c}{3} \log \left( \frac{l}{\epsilon} \right) + \text{const.},$$ (2.21)

where the last term is a non-universal constant and $l$ is the length of the interval. The central charge $c$ is a measure for the degrees of freedom in the theory. Since in two dimensions, the area of the boundary is the factor 2, this contains the expected area term for the UV cut-off. Using conformal mappings, this result can be generalized to non-vanishing temperature $T = 1/\beta$, yielding

$$S_{EE} = \frac{2c}{3} \log \left( \frac{\beta}{\pi \epsilon} \sinh \left( \frac{\pi l}{\beta} \right) \right) + \text{const.}.$$ (2.22)

Expanding this result in the large-volume limit or equivalently high-temperature limit yields

$$S_{EE} = \frac{2c}{3} \log \left( \frac{\beta}{\pi \epsilon} \right) + \frac{2c\pi l}{3\beta} + O(l^{-1}) + \text{const.}.$$ (2.23)

This contains the appropriate volume term for the thermal entropy. In a theory with a finite correlation length $\xi$, long-range correlation decrease exponentially. For $\xi \ll l$, the entanglement entropy becomes independent of the volume of the region (in two dimensions $l$) and only contains an area term (in two dimensions a constant).

$$S_{EE} = \frac{2c}{3} \log \left( \frac{\xi}{\epsilon} \right) \quad \xi \ll l$$ (2.24)

Comparison to the result for non-vanishing temperature shows that, after deducing the thermal contribution, the remaining entanglement entropy contains a constant term, which is an area law for a finite correlation length $\xi = 1/(\pi T)$.

Using the entanglement entropy, it is possible to define an entropic c-function. A c-function $C$ is a function which depends on the renormalized coupling constants and satisfies three properties. Firstly, it is non-increasing along the renormalization group (RG) flow. Secondly, the c-function is stationary at the fixed points of the RG
flow. A stationary $c$-function implies that all $\beta$ functions vanish and the theory is conformal. Lastly, the value of the $c$-function at the fixed points coincides with the central charge of the conformal theory describing the fixed point. Since the central charge is proportional to the degrees of freedom in a conformal theory, the $c$-function can be considered to be a measure of the degrees of freedom for a general field theory. For two dimensions, there is a function which satisfies these conditions [45], but this is not proven for higher dimensions (see e.g. [46]). For two-dimensional theories, the $c$-function can be calculated by

$$C = l \frac{dS_{EE}}{dl}, \quad (2.25)$$

where $S_{EE}$ is the entanglement entropy of an interval of length $l$. There are many attempts to generalize this to higher dimensions. However, many are complicated to calculate. Following [20], we will use the generalization

$$C = \frac{l^{d-1}}{l^{d-2}} \frac{dS_{EE}}{dl}, \quad (2.26)$$

where $S_{EE}$ is the entanglement entropy for a straight belt with width $l$. This function is dimensionless and cut-off independent, because the cut-off dependent terms only depend on the boundary (see equation (2.18)). Furthermore, it is easy to calculate from the entanglement entropy.

2.1.3 Thermodynamics

As explained above, the entanglement entropy is a measure for the missing information for an observer in the region $B$. At finite temperature, there are two contributions. The first one is the missing information about the complement $C$ and is due to its entanglement with $B$. Additionally, we have to consider that the system is no longer in a pure state. It is described by thermodynamics, where a complicated many-particle system is characterized by a few macroscopic variables, which do not fix the microstate uniquely. Therefore, the entanglement entropy also contains the thermal entropy, which is associated to the missing information about the microstate. The thermal entropy dominates the entanglement entropy in the large-volume limit. This enables us to calculate the thermal entropy and thermodynamical potentials from the entanglement entropy. Thermodynamical potentials can have a physical interpretation (e.g. the average energy $\langle E \rangle$) or give conclusions about phase transitions (e.g. the free energy $F$).

A system at fixed particle number, fixed volume and fixed temperature $T = 1/\beta$ is described by the canonical ensemble. The density matrix of the system is

$$\hat{\rho}_{can} = \frac{\exp(-\beta \hat{H})}{Z_{can}}, \quad (2.27)$$
where $\hat{H}$ is the Hamiltonian operator and $Z_{\text{can}}$ the partition function

$$Z_{\text{can}} = \sum_n \exp(-\beta E_n),$$

$$= \text{Tr} \exp(-\beta \hat{H}). \quad (2.28)$$

The expectation value for operators is defined as

$$\langle \mathcal{O} \rangle_{\text{can}} = \text{Tr} (\hat{\rho}_{\text{can}} \mathcal{O}). \quad (2.29)$$

One physical important expectation value is the average energy

$$\langle E \rangle = \langle \hat{H} \rangle_{\text{can}} = \text{Tr}(\hat{\rho}_{\text{can}} \hat{H}) = -\partial_\beta \ln Z_{\text{can}}. \quad (2.30)$$

It can be used to determine the time component of the stress energy tensor, which is the energy density. Furthermore, the diagonal spatial components are the pressure $P$, which is the negative density of the free energy $F$

$$F = -T \ln Z_{\text{can}},$$

$$= -PV. \quad (2.31a)$$

The free energy is the thermal potential of the canonical ensemble. The von Neumann entropy is defined as

$$S = \langle -\ln \hat{\rho} \rangle_{\text{can}} = -\text{Tr}(\hat{\rho}_{\text{can}} \ln \hat{\rho}_{\text{can}}),$$

$$= -\ln \hat{\rho}_{\text{can}} + \beta \text{Tr} \left( \hat{\rho}_{\text{can}} \hat{H} \right). \quad (2.32a)$$

It is a measure for the missing information about the microstate. The relationship between $F$, $\langle E \rangle$ and $S$ is

$$F = \langle E \rangle - TS,$$

$$dF = -SdT - PdV + \mu dN, \quad (2.33a)$$

where $P$ is the pressure and $\mu$ is the chemical potential.

One interesting aspect of thermodynamics are phase transitions. They occur when the thermal potential is non-analytic at a point. This can happen when there are two competing macroscopic states and macroscopic variables determine which one is stable. There are different kinds of phase transitions:

- first order phase transition: discontinuity of $F'$,
- second order phase transition: discontinuity of $F''$,
- ...
- $n^{\text{th}}$ order phase transition: discontinuity of $F^{(n)}$. 


2. Preliminaries: AdS/CFT

2.2 Gravity

Let us turn to the gravity side of the duality. The action for a spacetime is given by the Einstein Hilbert action

\[ S_{EH} = \frac{1}{16\pi G_N} \int d^Dx \sqrt{-g} (R - 2\Lambda), \]  

(2.34)

where \( G_N \) is the D-dimensional Newton’s constant and \( \Lambda \) is the cosmological constant. Minimizing this action yields the Einstein field equations

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \]  

(2.35a)

\[ = -\Lambda g_{ab}, \]  

(2.35b)

where \( G_{ab} \) is the Einstein tensor. In the presence of a matter contribution to the action, the Einstein field equations are sourced by the energy momentum tensor \( T_{ab} \)

\[ 8\pi G_N T_{ab} = G_{ab} + \Lambda g_{ab}, \]  

(2.36a)

\[ T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{ab}}. \]  

(2.36b)

Taking the trace yields the scalar curvature \( R \) in terms of \( T \) and \( \lambda \). This can be used to rewrite the Einstein field equations as

\[ R_{ab} = 8\pi G_N \tilde{T}_{ab} + \frac{2}{D-2} g_{ab} \Lambda, \]  

(2.37)

\[ \tilde{T}_{ab} = T_{ab} - \frac{1}{D-2} g_{ab} T^{cd} g_{cd}. \]  

(2.38)

\( \tilde{T}_{ab} \) is called the trace-reversed tensor.

2.2.1 AdS spacetime

For AdS/CFT, the Anti-de Sitter (or short AdS) spacetime is used on the gravity side. It is a vacuum solution (i.e. \( T_{ab} = 0 \)) to the Einstein field equations with negative cosmological constant \( \Lambda \)

\[ R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0. \]  

(2.39)

For any \( (d + 1) \) dimensional solution of this equation, we immediately obtains for the scalar curvature \( R \) and the Ricci tensor \( R_{ab} \) as

\[ R = \frac{d+1}{d-1} 2\Lambda, \]  

\[ R_{ab} = \frac{1}{d-1} 2\Lambda g_{ab}. \]  

(2.40)

Hence, AdS is a spacetime with constant negative curvature. Furthermore, the AdS spacetime is a maximally symmetric spacetime (i.e. it has the maximal number of symmetries). Because of this high symmetry, the Riemann tensor is uniquely determined to be

\[ R_{abcd} = \frac{R}{(d+1)d} (g_{bd} g_{ac} - g_{ad} g_{bc}). \]  

(2.41)
An intuitive approach to introduce $D = d + 1$ dimensional AdS spacetime is by embedding it into $d + 2$-dimensional Minkowski spacetime $\mathbb{R}^{d,2}$ with signature $(-,+,\ldots,+,−)$. The AdS spacetime is the sphere with radius $L$.

$$-(X^0)^2 + \sum_{i=1}^{n}(X^i)^2 - (X^{d+1})^2 = L^2. \quad (2.42)$$

Figure 2.2 shows the embedding of AdS$_2$ in $\mathbb{R}^3$. This space is invariant under rotation in the higher dimensional spacetime, which corresponds to the group $O(d,2)$. This group is $(d+1)(d+2)/2$ dimensional. This is the maximal number of independent symmetries of an $d+1$ dimensional spacetime.

A common choice of coordinates are the Poincaré patch coordinates $(r,t,\vec{x})$, which are defined by

$$X^0 = \frac{L^2}{2r} \left(1 + \frac{r^4}{L^4}(\vec{x}^2 - t^2 + L^2)\right), \quad (2.43a)$$

$$X^i = \frac{rx^i}{L} \text{ for } i \in 1,\ldots,d-1, \quad (2.43b)$$

$$X^d = \frac{L^2}{2r} \left(1 + \frac{r^4}{L^4}(\vec{x}^2 - t^2 - L^2)\right), \quad (2.43c)$$

$$X^{d+1} = \frac{rt}{L}. \quad (2.43d)$$

The metric in this coordinates is

$$ds^2 = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} \left(-dt^2 + d\vec{x}^2\right) \quad (2.44)$$

and is singular at $r = 0$. Hence, the new coordinates are restricted to $r > 0$ and cover only one half of the complete AdS spacetime. The other half is covered by the same
coordinates with \( r < 0 \). A similar coordinate choice is \( z = L^2/r \). The metric in this coordinates is

\[
ds^2 = \frac{L^2}{z^2} \left( dz^2 - dt^2 + dx^2 \right) = \frac{L^2}{z^2} \left( dz^2 + \eta_{\nu\mu} dx^\nu dx^\mu \right),
\]

(2.45)

where we use Einstein sum convention and sum over \( \mu = 1, \ldots d \).

### 2.2.2 Asymptotically AdS

We also consider less symmetric solutions of equation (2.39) in the AdS/CFT duality, which are only asymptotically AdS. This means that the metric can be written as

\[
ds^2 = \frac{L^2}{z^2} \left( dz^2 + \hat{g}_{\nu\mu}(z, x) dx^\nu dx^\mu \right),
\]

(2.46)

which generalizes equation (2.45). This form of the metric is called Fefferman-Graham form. The metric \( \hat{g}_{\nu\mu} \) has to be finite for \( z \to 0 \). An asymptotic AdS metric is singular at the boundary. However, it is still possible to define a boundary metric by using a defining function \( \omega(z, t, \vec{x}) \). If \( \omega \) is smooth, positive and has a second order zero at \( z = 0 \), a boundary metric can be defined by

\[
ds^2_{\partial \text{AdS}} = \lim_{z \to 0} \omega(z, t, \vec{x}) ds^2 |_{z = \text{const.}}.
\]

(2.47)

This boundary metric depends on the choice of \( \omega \). Therefore, the bulk metric only determines the boundary metric up to conformal transformations. The reason for this lies in the gauge freedom due to diffeomorphism invariance. A specific transformation of the metric is the Penrose-Brown-Henneaux transformation

\[
\begin{align*}
z &= \tilde{z} \left( 1 - 2 \sigma(\tilde{x}) \right), \\
x^\mu &= \tilde{x}^\mu + a^\mu(\tilde{x}, \tilde{z}),
\end{align*}
\]

(2.48a, 2.48b)

with \( \partial_{\tilde{z}} a^\mu = z L^2 g^{\mu\nu} \partial_{\tilde{z}} \sigma(x) \) and \( a_\mu(x, 0) = 0 \). It keeps a metric in the Fefferman-Graham form and reduces at the boundary to an infinitesimal Weyl transformation

\[
\hat{g}_{\mu\nu}(0, x) \to (1 + 2 \sigma(x)) \hat{g}_{\mu\nu}(0, x).
\]

(2.49)

For this reason, the boundary of an asymptotically AdS spacetime is called conformal.

One example of an asymptotically AdS metric is the five dimensional AdS-Schwarzschild metric

\[
\begin{align*}
ds^2_{\text{AdS}} &= \frac{L^2}{z^2} \left( \frac{dz^2}{b(z)} - b(z) dt^2 + d\vec{x}^2 \right), \\
b(z) &= 1 - \frac{z^4}{z_h^4}.
\end{align*}
\]

(2.50a, 2.50b)
Using the transformation \( z = \tilde{z}/\sqrt{1 + \frac{\tilde{z}^4}{4\tilde{z}_h^4}} \), this can be brought to the Fefferman-Graham form with

\[
\tilde{g}_{\nu\mu}(\tilde{z}, x^\sigma) dx^\nu dx^\mu = -\frac{(1 - \frac{\tilde{z}^4}{4\tilde{z}_h^4})^2}{1 + \frac{\tilde{z}^4}{4\tilde{z}_h^4}} dt^2 + (1 + \frac{\tilde{z}^4}{4\tilde{z}_h^4}) dx^i dx^i.
\] (2.51)

This spacetime has an event horizon at \( z = z_h \) and contains a black hole. Expanding the metric (2.50) in Euclidean signature around the horizon yields

\[
b(z) \approx 4 \frac{z_h - z}{z_h},
\] (2.52a)

\[
ds_{AdS}^2 \approx \frac{L^2}{z_h^2} \left( \frac{dz^2}{4} - \frac{z_h - z}{z_h} d\tau^2 + dx^2 \right),
\] (2.52b)

where \( \tau = it \) is the imaginary time. Changing the coordinates to

\[
r = L \sqrt{1 - \frac{z}{z_h}},
\] (2.53a)

\[
\varphi = \frac{2}{z_h} \tau
\] (2.53b)

brings the metric to the form

\[
ds_{AdS}^2 \approx dr^2 + r^2 d\varphi^2 + \cdots.
\] (2.54)

To avoid the conical singularity, \( \varphi \) has to be periodic with period \( 2\pi \). In quantum statistical mechanics, the corresponding period of the imaginary time \( \beta = z_h \pi \) is identified with the inverse temperature

\[
T = \frac{1}{\beta z_h} = \frac{1}{\pi z_h}.
\] (2.55)

This is the Hawking temperature of the black hole.

The entropy of the black hole can be calculated by using the Bekenstein-Hawking entropy.

\[
S = \frac{1}{4G_N} \text{Area}_{\text{Horizon}}.
\] (2.56)

In the AdS\(_5\) case, this results in

\[
S = \frac{1}{4G_N} \pi^3 L^3 T^3 \text{Vol}(\mathbb{R}^3).
\] (2.57)

The free energy \( F \) and the average energy \( \langle E \rangle \) can be calculating by using equations (2.33).

\[
F = -\frac{1}{16G_N} \pi^3 L^3 T^4 \text{Vol}(\mathbb{R}^3)
\] (2.58a)

\[
\langle E \rangle = \frac{3}{16G_N} \pi^3 L^3 T^4 \text{Vol}(\mathbb{R}^3)
\] (2.58b)
2. Preliminaries: AdS/CFT

See also [47] for the results for general dimension.\(^6\)

This thermodynamical analysis can also be performed by calculating the (regularized) Euclidian on-shell action \(\mathcal{I}\).\(^7\) The free energy is then given by (2.31a)

\[
F = - T \ln Z = T \mathcal{I}.
\]  

(2.59)

2.3 String Theory

This section reviews the concept of bosonic string theory. Later on, in section 2.3.2 this discussion is generalized to superstring theory.

2.3.1 Bosonic string theory

Strings are one-dimensional objects in a \(D\) dimensional target space. They are parametrized by the worldsheet coordinates \((\tau, \sigma)\), where \(\tau\) is the proper time and \(\sigma\) is the spatial extent. Analogous to a point particle, where the action is simply the proper time, a bosonic string can be described by the Nambu-Goto action

\[
S_{NG} = - \frac{1}{2\pi \alpha'} \int_{\Sigma} d\tau d\sigma \sqrt{-\det(\partial_\alpha X^M \partial_\beta X^N \eta_{MN})},
\]  

(2.60)

which is classically equivalent to the Polyakov action

\[
S_P = - \frac{1}{4\pi \alpha'} \int_{\Sigma} d\tau d\sigma \sqrt{-h^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \eta_{MN}}.
\]  

(2.61)

The worldsheet metric \(h_{\alpha\beta}(\tau, \sigma)\) is introduced as an auxiliary field. Its equation of motion is

\[
T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta} \gamma^\gamma \partial_\gamma X \cdot \partial_\delta X = 0.
\]  

(2.62)

As a result of reparametrization invariance and Weyl invariance in the worldsheet coordinates, it can always be chosen the conformal gauge where \(h\) is the two-dimensional Minkowski metric. In this gauge, the action and the energy momentum tensor are

\[
S_P = \frac{1}{4\pi \alpha'} \int_{\Sigma} d\tau d\sigma (\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X),
\]  

(2.63)

\[
T_{\alpha\beta} = \eta_{\alpha\beta} (\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_\tau X \cdot \partial_\sigma X.
\]  

(2.64)

---

\(^6\)The spacetime coordinates are scaled differently in this paper. Therefore, the volume term obtains an additional factor \(\text{Vol}(M^{d-2}) = \text{Vol}(\mathbb{R}^{d-2})/L^{d-2}\).

\(^7\)Since the AdS space has a boundary, we also have to consider the Gibbons-Hawking boundary term

\[
S_{GH} = \frac{1}{8\pi G_N} \int d^D x \sqrt{-\gamma} K,
\]

where \(\gamma\) is the induced metric on the boundary and \(K\) is the trace of the extrinsic curvature.
2.3 String Theory

The range of $\sigma$ is finite $\sigma \in [0, \sigma_{\text{max}}]$. Besides the equation of motion, minimizing the action also yields the boundary condition

$$\left[\partial_\sigma X^M \delta X_M\right]_{\sigma_{\text{max}}}^0 = 0.$$  \hfill (2.65)

This is trivially satisfied for closed strings with $X^M(\sigma) = X^M(\sigma + \sigma_{\text{max}})$. For open strings with endpoints $\sigma_0 \in \{0, \sigma_{\text{max}}\}$, this boundary term vanishes in two different cases:

- Neumann boundary condition: $\partial_\sigma X^M(\tau, \sigma_0) = 0$
  
  These boundary conditions ensure that there is no momentum flow at the endpoint.

- Dirichlet boundary condition: $\partial_\sigma X^M(\tau, \sigma_0) = 0$
  
  This boundary condition fixes the endpoint of the string.

This boundary conditions can be chosen for every endpoint and every coordinate direction independently. The combinations of the boundary conditions is then called NN, DD, DN or ND.

In the conformal gauge, the equation of motion for $X$ simplifies to

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)X^M = \partial_\tau \partial_- X^\mu = 0,$$  \hfill (2.66)

$$\sigma^\pm = \tau \pm \sigma,$$  \hfill (2.67)

which allows to decompose the excitations in left- and right-movers. Choosing $\sigma_{\text{max}}$ appropriately\(^8\) allows the following ansatz for all boundary conditions

$$X^M(\tau, \sigma) = X^M_{(L)}(\sigma^+) + X^M_{(R)}(\sigma^-),$$  \hfill (2.68a)

$$X^M_{(L)}(\sigma^+) = \frac{x^M_0}{2} + \sqrt{\frac{\alpha'}{2}} \tilde{\alpha}^M_0 \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \tilde{\alpha}^M_n \frac{n}{n} \exp(-in\sigma^+),$$  \hfill (2.68b)

$$X^M_{(R)}(\sigma^-) = \frac{x^M_0}{2} + \sqrt{\frac{\alpha'}{2}} \alpha^M_0 \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha^M_n \frac{n}{n} \exp(-in\sigma^-),$$  \hfill (2.68c)

where reality of $X$ demands $\alpha^M_{-n} = (\alpha^M_n)^*$, $\tilde{\alpha}^M_{-n} = (\tilde{\alpha}^M_n)^*$ and the boundary conditions give restrictions on the Fourier coefficients. Using canonical quantization, the commutation relations for $\alpha$ and $\tilde{\alpha}$ decouple and become

$$[\alpha^M_m, \alpha^N_n] = m \eta^{MN} \delta_{m+n,0},$$  \hfill (2.69a)

$$[\tilde{\alpha}^M_m, \tilde{\alpha}^N_n] = m \eta^{MN} \delta_{m+n,0}.$$  \hfill (2.69b)

\(^8\)The excitations of the string are standing waves. For closed strings, it is convenient to choose $\sigma_{\text{max}} = 2\pi$. Since the Dirichlet boundary condition is a fixed endpoint and the Neumann boundary condition is an open one, it is convenient to choose $\sigma_{\text{max}} = \pi$ for the configurations with the same boundary conditions on both endpoints and $\sigma_{\text{max}} = \pi/2$ for configurations with different boundary condition at both endpoints.
This shows, that the spatial components of $\alpha$ and $\tilde{\alpha}$ are proportional to ladder operators $a$ and $\tilde{a}$ of an harmonic oscillator. However, the time components produce negative norm states. Luckily, they decouple from the remaining states.

Using this ansatz, the vanishing of the energy momentum tensor can be written as

$$L_m = \frac{1}{2} \sum_n : \alpha_n \cdot \alpha_{m-n} : = a_m, \quad (2.70a)$$

$$\tilde{L}_m = \frac{1}{2} \sum_n : \tilde{\alpha}_n \cdot \tilde{\alpha}_{m-n} : = \tilde{a}_m, \quad (2.70b)$$

where the constants $a_m$ have to be introduced due to the normal ordering $: \cdot :$. Without normal ordering, applying this operator on a physical state yields zero. Since only $L_0$ and $\tilde{L}_0$ are affected by normal ordering, only $a_0 = \tilde{a}_0$ is non-vanishing. Hence, this gives the condition

$$L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \tilde{N} = a, \quad (2.71a)$$

$$\tilde{L}_0 = \frac{1}{2} \tilde{\alpha}_0 \cdot \tilde{\alpha}_0 + N = a, \quad (2.71b)$$

with number operators $N$ and $\tilde{N}$. For a closed string, the periodicity requires $\alpha_0 = \tilde{\alpha}_0$, which causes level matching: $N = \tilde{N}$. Using the canonical momentum, the mass of a string is

$$M^2 = \frac{1}{\alpha'} (N - a) \cdot \begin{cases} 1 & \text{open string, NN} \\ 4 & \text{closed} \end{cases}. \quad (2.72)$$

The problem with the negative norm states produced by $\alpha_0$ and $\tilde{\alpha}_0$ can be solved by gauge fixing, analogously to the approach in electrodynamics. After gauge-fixing, the normal ordering constant $a$ is

$$a = \frac{1}{2} (D - 2) \sum_{n=1}^{\infty} n = -\frac{D - 2}{24}, \quad (2.73)$$

where $D$ is the dimension of the target spacetime. The first excited state for open strings is

$$\alpha^1_1|0, k\rangle : \text{vector boson}, \quad (2.74)$$

which transforms under $SO(D-2)$. To avoid anomalies in the Lorentz symmetry, this state has to be a massless. This fixes $a = 1$ and the dimension of the target spacetime $D = 26$. The first excited state of the closed string sector is also massless.

$$\alpha^1_1 \tilde{\alpha}^2_1|0, \vec{k}\rangle : \text{rank 2 tensor}. \quad (2.75)$$

This state can be decomposed into irreducible representations of $SO(D-2)$ by splitting of the traceless symmetric, the antisymmetric and the trace part.
2.3 String Theory

2.3.2 Superstring theory

Bosonic string theory only describes bosonic degrees of freedom. One way to achieve fermionic degrees of freedom is to have worldsheet supersymmetry. The quantization and the calculation of the spectrum is then analogous to bosonic string theory.

For every bosonic field $X^M$, a fermionic superpartner $\Psi^M$ is introduced. The fermionic part of the action is

$$ S_f = \frac{1}{4\pi\alpha'} \int d^2\sigma \bar{\Psi}^M \rho^a \partial_a \Psi^M, $$

$$ = \frac{-i}{2\pi\alpha'} \int d^2\sigma \left( \bar{\psi}_+ \partial_+ \psi_- + \bar{\psi}_- \partial_+ \psi_+ \right). $$

The equation of motion for the fermions

$$ \partial_+ \psi_- = \partial_- \psi_+ = 0 $$

yields a solution in terms of left- and right-moving waves. The ansatz for the mode expansion is

$$ \psi_+ = \sqrt{\frac{1}{2}} \sum_n d_n \exp(-i n \sigma_+), \quad \psi_- = \sqrt{\frac{1}{2}} \sum_n \tilde{d}_n \exp(-i n \sigma_-), $$

where reality demands $d_{-n} = (d_n)^*$, $\tilde{d}_{-n} = (\tilde{d}_n)^*$. Using the same method to quantize the theory as in the bosonic part, the anti-commutation relations for the Fourier coefficients are

$$ \{d^M_m, d^N_n\} = \eta^{NM} \delta_{m+n,0}. $$

For the spatial components, this corresponds creation and annihilation operators of quantum harmonic oscillators. For the time component, the anti-commutator has the wrong sign, which causes negative norm states. A special case of the algebra is $m = n = 0$. The algebra then simplifies to the Clifford algebra. Therefore, $\sqrt{2}d_0$ is a Dirac matrix acting on the states. If $d_0 \neq 0$ is allowed by the boundary conditions, every state is a fermion, otherwise every state is a boson. The boundary conditions have to satisfy

$$ [\psi_+ \delta \psi_+ - \psi_- \delta \psi_-]_{\sigma=0} = 0 $$

For open strings, the contributions from both endpoints have to vanish independently. This gives $\psi_+ (\tau, 0) = \psi_- (\tau, 0)$, where the sign is a matter of convention. For the other

---

$^9$ Used conventions:

$$ \rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} $$
boundary $\sigma_{\text{max}} = \pi$, there are two different boundary conditions, depending on the relative sign:

\[
\begin{align*}
\text{Neveu Schwarz boundary condition} & \quad \psi_+ (\tau, \pi) = \psi_- (\tau, \pi), \\
\text{Ramond boundary condition} & \quad \psi_+ (\tau, \pi) = -\psi_- (\tau, \pi).
\end{align*}
\]

This boundary condition gives restrictions on the Fourier coefficients, which yield that the Neveu-Schwarz sector contains bosons, whereas the Ramond sector contains fermions. For closed strings, both left- and right-movers are chosen from the Ramond or Neveu-Schwarz sector independently. Therefore, there are four different solutions, labeled depending on the boundary conditions as R-R, NS-NS, R-NS and NS-R.

Repeating the same procedure as in the bosonic case, the same mass formula as in equation (2.72) is obtained after redefining number operator $N$

\[
N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{n=1}^{\infty} d_{-n} \cdot d_n
\]

and the constant $a$

\[
a = \begin{cases} 
\frac{1}{2} & \text{NS sector} \\
0 & \text{R sector}
\end{cases}.
\]

Lorentz invariance requires $D = 10$.

There are still problems with this solution. The vacuum state of the Neveu-Schwarz sector is a tachyon. Furthermore, the closed string spectrum contains a gravitino. Therefore, the theory is only consistent if there is spacetime supersymmetry, which requires an equal number of fermionic and bosonic degrees of freedom for each mass.

This is solved by using the GSO projection. For the Neveu-Schwarz sector, we only take the states where $\sum_{n>0} d_{-n} d_n$ is odd. This gets rid of the tachyon. For the Ramond sector, we can consider either only states with the same chirality for left- and right-movers or only states with the different chirality. The first method produces type IIB superstring theory and the second one type IIA superstring theory. For type IIB superstring theory, the massless open string spectrum consists of $d_{-1/2}[0]_{\text{NS}}$ and $|+\rangle_R$. The massless closed string spectrum is obtained by splitting the states into irreducible $SO(8)$ representations and is shown in Table 2.3.

### 2.3.3 Supergravity

One interesting limit of closed superstring theory is the low energy limit. In this limit, all massive modes are integrated out and only the massless modes remain. Hence, the particle content of this effective theory is shown in Table 2.3. The equation of motion of this theory are equivalent to the equation of motions of the supergravity (or short SUGRA) action. The bosonic part of this action is

\[
S_{\text{IIB}} = \frac{1}{2 \kappa_{10}^2} \left[ \int d^{10} X \sqrt{-g} \left( e^{-2\phi} \left( R + 4 \partial M \partial^M \phi - \frac{1}{2} |H_3|^2 \right) \\
- \frac{1}{2} |F_1|^2 - \frac{1}{2} |F_3|^2 - \frac{1}{4} |F_5|^2 \right) + \frac{1}{2} \int C_4 \wedge H_3 \wedge F_3 \right],
\]

\[\text{(2.84)}\]
2.3 String Theory

<table>
<thead>
<tr>
<th>Sector</th>
<th>Field</th>
<th>Particle</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS-NS</td>
<td>$\phi$</td>
<td>scalar (dilaton)</td>
</tr>
<tr>
<td></td>
<td>$B_{(2)}$</td>
<td>antisymmetric two-form gauge field (Kalb-Ramond field)</td>
</tr>
<tr>
<td></td>
<td>$g_{MN}$</td>
<td>metric (graviton)</td>
</tr>
<tr>
<td>R-R</td>
<td>$C_{(n)}$, $n \in {0, 2}$</td>
<td>$n$-form gauge field</td>
</tr>
<tr>
<td></td>
<td>$C_{(4)}$</td>
<td>self-dual 4-form gauge field</td>
</tr>
<tr>
<td>NS-R</td>
<td>$\lambda_I$, $I \in {1, 2}$</td>
<td>spin 1/2 dilatinos</td>
</tr>
<tr>
<td></td>
<td>$\Psi^M_I$, $I \in {1, 2}$</td>
<td>spin 3/2 gravitinos with same chirality</td>
</tr>
</tbody>
</table>

Table 2.3: Particle content of IIB string theory

where the conventions

$$F_{(p)} = dC_{(p-1)}, \quad H_{(3)} = dB_{(2)}, \quad \tilde{F}_{(3)} = F_{(3)} - C_{(0)}H_{(3)}$$

(2.85)

are used. For supergravity, it is necessary to demand the self-duality of $F_{(5)}$ separately. To obtain the canonical normalised action, we have to take care of the vacuum expectation value of the dilaton $\phi_0$. The relationship between gravitational constant $2\kappa_{10}^2 = (2\pi)^7\alpha'^4$ and the ten-dimensional Newton’s constant $G$ is

$$2\kappa_{10}^2e^{2\phi_0} = 2\kappa_{10}^2 = 16\pi G_{10}.\quad (2.86)$$

The vacuum expectation value $\phi_0$ is also important in string perturbation theory. In the worldsheet action, $\phi_0$ appears as prefactor of the Euler number. This is a topological term and connected to the number of handles of a worldsheet. Hence, in the path integral approach, contributions from different topologies will be added with weight $e^{\phi_0\xi}$. A handle in the worldsheet corresponds to emission and absorption of a string. Therefore, this defines the string coupling constant $g_s$ as

$$g_s = e^{\phi_0}.\quad (2.87)$$

The gravitational constant is therefore

$$\kappa_{10} = \frac{1}{\sqrt{2}}\sqrt{2\pi^7\alpha'^2}g_s.\quad (2.88)$$

2.3.4 D-branes

The origin of the AdS/CFT is a duality between different descriptions of certain objects, called D-branes. They can be understood in the open and in the closed string picture.

For open strings, we saw above that there are two different boundary conditions: Neumann boundary conditions and Dirichlet boundary conditions. For Dirichlet boundary conditions, the endpoints of the string are fixed and there is a momentum flow at the endpoints. For a string with $p + 1$ spacetime directions with Neumann boundary conditions and $D - p - 1$ spatial directions with Dirichlet boundary conditions, the
endpoints have to lie on a \( p + 1 \) dimensional hypersurface. These hyperplanes are called D\( p \)-branes. Besides this open string perspective, we can also understand D\( p \)-branes in the closed string perspective, where they are massive objects. They are higher dimensional analogues to black holes with a \( p + 1 \) dimensional horizon and curve the space around them.

Let us start with reviewing the open string perspective, before moving on to the closed string perspective.

**Open string perspective**

In the low-energy limit, it is possible to describe the dynamics of open strings by a worldvolume action of the D\( p \)-brane. The bosonic part of this action is the Dirac-Born-Infeld action or short DBI action

\[
S_{\text{DBI}} = -\tau_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det \left( g_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta} \right)}. \tag{2.89}
\]

The massless bosonic fields of the brane are the worldvolume gauge field \( A_\mu \) and \( D-(p-1) \) goldstone modes from breaking the Lorentz symmetry in the transverse directions of the brane. The DBI action contains the coupling to the NS-NS sector of the massless closed string spectrum: the dilaton \( \phi \), the metric \( g \) and the Kalb-Ramond field \( B = B_{(2)} \). The indices \( \alpha \) and \( \beta \) are the worldvolume indices and \( g_{\alpha\beta} \) and \( B_{\alpha\beta} \) are the pullback of the bulk fields \( g_{ab} \) and \( B_{ab} \). The prefactor of the DBI action is

\[
\tau_p = (2\pi)^{-(p-1)/2}. \tag{2.90}
\]

Some simple cases help to understand the meaning of the DBI action. If only the metric \( g \) and the dilaton \( \phi = \ln g_s = \text{const.} \) are non-vanishing, the action simplifies to the volume of the D\( p \)-brane

\[
F = B = 0, \quad e^\phi = g_s : \quad S_{\text{DBI}} = -\frac{\tau_p}{g_s} \int d^{p+1}\xi \sqrt{-\det g_{\alpha\beta}}. \tag{2.91}
\]

The prefactor \( \tau_p/g_s \) is the tension of the brane. The scaling \( 1/g_s \) makes D\( p \)-branes to non-perturbative objects. Expanding the DBI action for small field strength in flat spacetime reduces the DBI action to the Yang-Mills action

\[
B = 0, \quad e^\phi = g_s : \quad S_{\text{DBI}} = \text{const.} - \frac{\tau_p}{4g_s} (2\pi\alpha')^2 \int d^{p+1}\xi F^{\alpha\beta} F_{\alpha\beta} \tag{2.92}
\]

Therefore, the DBI action is a non-linear generalization of Yang-Mills theory. The corresponding Yang-Mills coupling is

\[
g_{YM}^2 = (2\pi)^{p-2} g_s \alpha' \frac{\tau_p}{4}. \tag{2.93}
\]

This implies that the coupling is in general dimension-full. The effective coupling constant at an energy scale \( E \) is

\[
g_{\text{eff}}^2(E) \sim g_{YM}^2 E^{p-3}. \tag{2.94}
\]
The case \( p = 3 \) is special, since the coupling constant is dimensionless and the effective coupling is energy-independent.

The \( D_p \) brane also couples to the R-R sector gauge fields. In analogy to the coupling between a charged point particle and an electromagnetic field, the most trivial coupling is

\[
S_p = \frac{\tau_p}{g_s} \int_{\Sigma_{p+1}} P[C_{(p+1)}].
\]  

(2.95)

The complete coupling to all gauge fields is described by the Chern-Simons term

\[
S_{CS} = \frac{\tau_p}{g_s} \int_{\Sigma_{p+1}} \sum_q P[C_{(q)}] \wedge \exp^{P[B]+2\pi\alpha'}F,
\]  

(2.96)

where the integral picks the appropriate \((p + 1)\) form.

When considering a stack of \( N D_p \)-branes, there are string stretching between different branes as shown in Figure 2.3. This gives every open-string state the multiplicity \( N^2 \). This promotes the gauge symmetry from \( U(1) \) to \( U(N) \) and all open-string states are in the adjoint representation. The DBI action generalizes to

\[
S_{DBI} = -\tau_p \int d^{p+1}x e^{-\phi} \text{Tr} \sqrt{-\det (g_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha'F_{\alpha\beta})},
\]  

(2.97)

where \( g \) and \( B \) are multiplied by unit matrices and the expression inside the trace is evaluated for each of the \( N^2 \) components separately. For the case that only the \( U(1) \) field strength is non-vanishing, this reproduces the DBI action for a single \( D_p \) brane multiplied by a factor of \( N \).

**Closed string perspective**

In the closed string perspective, \( D_p \)-branes can be described as black \( p \)-branes. Black \( p \)-branes are higher dimensional analogues to black holes. Whereas the singularity
is point-like in four dimensions, it can be \( p \)-dimensional in higher dimensions, which breaks the Lorentz invariance down to

\[
SO(d, 1) \rightarrow SO(d - p) \times SO(p, 1).
\] (2.98)

Dp-branes are black \( p \)-branes which appear as solitonic solution of SUGRA. In ten dimensions, the coordinates can be split into \( p + 1 \) coordinates along the brane \( x^\mu \) and \( 9 - p \) coordinates perpendicular to the brane \( y^i \). Qualitatively, there are two different solutions. Extremal black \( p \)-branes are solutions which preserve the maximal number of supersymmetries. In contrast, non-extremal black \( p \)-branes are non-supersymmetric.

The ansatz for the extremal solution is

\[
ds^2 = H_p(r)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_p(r)^{1/2} dy^i dy^i,
\] (2.99a)

\[
e^\phi = g_s H_p(r)^{(3-p)/4},
\] (2.99b)

\[
C_{(p+1)} = \left( H_p(r)^{-1} - 1 \right) dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p,
\] (2.99c)

\[
B_{MN} = 0,
\] (2.99d)

where \( r^2 = \sum (y^i)^2 \). Solving this ansatz for \( H_p \) yields

\[
\Box H_p(r) = 0,
\] (2.100)

\[
H_p(r) = 1 + \left( \frac{L_p}{r} \right)^{7-p}.
\]

The integration constant is chosen in such a way that we obtain asymptotically Minkowski space at \( r \to \infty \). For \( r \to 0 \), \( H_p \) diverges. Therefore, the metric is singular and we have a \( p + 1 \)-dimensional singularity.

This solutions are charged under the \( C_{(p+1)} \). The corresponding charge is

\[
Q = \frac{1}{2\kappa_{10}^2} \int_{S^{8-p}} F_{(p+2)} = N \frac{\tau_p}{g_s},
\] (2.101)

where \( N \) is the total number of black \( p \)-branes. Calculating this charge yields

\[
L_p^{7-p} = (4\pi)^{(5-p)/2} \Gamma \left( \frac{7-p}{2} \right) g_s N \alpha'^{(7-p)/2}.
\] (2.102)

Of special interest is the case \( p = 3 \), where the dilaton is constant. Whereas other branes couple only magnetically or electrically to a \( C_{(n)} \)-form, the D3-branes couple both magnetically and electrically to the \( C_{(4)} \)-form. The corresponding field strength is self-dual and the brane carries a self-dual charge. The specific value for \( L_3 \) is

\[
L_3 = 4\pi g_s N \alpha'^2.
\] (2.103)

The constant dilaton corresponds to the fact that the Yang-Mills coupling \( g_{YM} \) is energy-independent.
For the non-extremal (i.e. non-supersymmetric) solutions, only the metric is different.

\[ ds^2 = H_p(r)^{-1/2} \left( -f(r)dt^2 + dx^i dx^i \right) + H_p(r)^{1/2} dy^i dy^i \]  

(2.104a)

\[ f(r) = 1 - \frac{r_{\text{h}}^7 - p}{r^7 - p} \]  

(2.104b)

Additionally to the inner horizon at \( r = 0 \), they also have an event horizon at \( r = r_{\text{h}} \). We obtain the extremal solution in the limit where both horizons are the same (i.e. \( r_{\text{h}} \to 0 \)). The reason for this is that black \( p \)-branes are higher dimensional analogues to Reissner-Nordström black holes. Non-extremal ones have an inner and an outer horizon, whereas extremal ones corresponds to the limit where both horizons coincide.

### 2.4 AdS/CFT

The previous section reviewed the key ingredients needed for the AdS/CFT correspondence. Finally, we can turn our attention to the duality, which realizes the holographic principle and gives a precise mapping between gravity theory and dual field theory.

#### 2.4.1 The AdS/CFT correspondence

The AdS/CFT correspondence was conjectured by Maldacena in 1997 [10]. The conjecture states that in the large \( N_c \) limit, certain conformal fields theories are dual to SUGRA on AdS space times a compact space. The motivation behind this idea is to compare the low-energy limit of branes in the open and the closed string perspective. Additionally to this paper and the already mentioned books, where [8] has the most detailed information, this section follows the early review [11].

In the following, we consider D3-branes and take the low-energy limit. Let us first take a look at the open string perspective, before turning our attention to the closed string perspective.

**Open string perspective**

Due to the conformal symmetry of string theory, the low-energy limit can be obtained by fixing the considered energy scale and taking the limit \( \alpha' \to 0 \). All massive states have a mass proportional to \( M^2 \propto 1/\alpha' \) and become heavy. The effective action is then obtained by integrating out all excitations except the massless ones. The effective action in the open string perspective is of the form

\[ S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}. \]  

(2.105)

\( S_{\text{brane}} \) contains the open and \( S_{\text{bulk}} \) contains the closed string states. Furthermore, interactions between the different string sectors are contained in \( S_{\text{int}} \).

Since the D3-branes preserve part of the supersymmetries, the massless states can be grouped into supermultiplets. The particle content of the massless open string
spectrum is a $\mathcal{N} = 4$ gauge multiplet. In the low-energy limit, $S_{\text{brane}}$ reduces to $\mathcal{N} = 4$ SYM theory on the four dimensional worldvolume with coupling $g_{YM}^2 = 2\pi g_s$. After canonical normalization, the interaction part $S_{\text{int}}$ is proportional to a positive power of $\alpha'$ and consequently vanishes in the low-energy limit. This means that open strings and closed strings decouple and can be described independently. The particle content of the massless closed string spectrum is a $\mathcal{N} = 1$ gravity multiplet in ten dimensions. In the low-energy limit, the closed string states are effectively described by free SUGRA in ten dimensions.

Summarizing this part, the resulting theory is

$$\mathcal{N} = 4 \text{ SYM theory on } \mathbb{R}^{3,1} + \text{ type IIB SUGRA on } \mathbb{R}^{9,1}.$$ 

Closed string perspective

In the closed string perspective, we are no longer in flat spacetime. The energies measured in the field theory are the energies measured by an observer at infinity. The low-energy limit is therefore

$$E_{\infty} = H_3(r)^{-1/4} E_r \ll 1/\sqrt{\alpha'}, \quad (2.106)$$

where $E_p$ is the energy measured at $r$ and $E_{\infty}$ is the energy as measured at infinity. There are two different types of excitations which satisfy this inequality. One type are massless bulk excitations (e.g. excitations with $E_p = 0$), which satisfy it independent of $r$. They can propagate in the whole space. The other type of excitations are near-horizon excitations which are located at $r/L \to 0$ and have a finite energy $E_p$. They cannot escape this near-horizon region. The wavelength of the bulk-excitations is redshifted in the near-horizon region and larger than the gravitational size of the brane ($\sim L$). Due to this, the cross-section between the different excitations goes to zero and the bulk excitations decouple from the near-horizon excitations.

The limit $r \to 0$ has to be taken carefully, because $L \sim \sqrt{\alpha'}$. To obtain arbitrary excitation in the near horizon region, the dimensionless quantity $\sqrt{\alpha'} E_p$ has to be fixed. The energy measured at infinity is

$$E_{\infty} \sim \frac{L}{\alpha'} \sqrt{\alpha'} E_p. \quad (2.107)$$

This is the energy measured in the field theory and should also kept fixed. Consequently, the appropriate limit is $\alpha' \to 0$, $r \to 0$ while keeping $U = r/\alpha'$ fixed.

Changing the coordinates and performing the $\alpha' \to 0$ limit yields

$$H_3(r) \approx \frac{4\pi g_s N}{U^4 \alpha'^2}, \quad (2.108a)$$

$$ds^2 \approx \alpha' \left( \frac{U^2}{\sqrt{4\pi g_s N}} \eta_{\mu\nu} dx^\mu dx^\nu + \sqrt{\frac{4\pi g_s N}{U^2}} \left( dU^2 + U^2 d\Omega_5^2 \right) \right). \quad (2.108b)$$

Naively, it looks as if the metric vanishes in this limit. However, the gravitational constant ($2.88$) is proportional to $\kappa_{10} \sim \alpha'^2$. The Lagrangian for SUGRA contains
\[ \sim \frac{1}{\kappa^2_{10}} \sqrt{-g} \] and consequently, \( \alpha' \) can be cancelled out. In the original coordinates, the result is

\[ ds^2 \approx \frac{r^2}{L^2} \eta_{\mu \nu} dx^\mu dx^\nu + \frac{L^2}{r^2} \left( dr^2 + r^2 d\Omega_5^2 \right). \tag{2.109} \]

After the coordinate transformation \( r = L^2 / z \), the near-horizon limit is

\[ ds^2 = \frac{L^2}{z^2} \left( dz^2 - dt^2 + dx^2 \right) + L^2 d\Omega_5^2. \tag{2.110} \]

The bulk excitations yield type IIB SUGRA in 10 dimensional Minkowski space. Combining both types of excitations, the theory is

\[ \text{type IIB SUGRA on } \text{AdS}_5 \times S^5 + \text{type IIB SUGRA on } \mathbb{R}^{9,1}. \]

**Duality**

Comparing this two perspectives yields the equivalence between ( \( \mathcal{N} = 4 \) SYM theory on \( \mathbb{R}^{3,1} + \text{type IIB SUGRA on } \mathbb{R}^{9,1} \)) and (type IIB SUGRA on \( \text{AdS}_5 \times S^5 + \text{type IIB SUGRA on } \mathbb{R}^{9,1} \)). Since both sides contain type IIB SUGRA on \( \mathbb{R}^{9,1} \), this conjectures the duality

\[ \mathcal{N} = 4 \ SU(N_c) \text{ SYM theory on } \mathbb{R}^{3,1} \leftrightarrow \text{type IIB superstring theory on } \text{AdS}_5 \times S^5. \]

Let us take a closer look on the limits taken above. In the large \( N_c \) limit of gauge theories, the effective coupling is the \('t Hooft coupling\)

\[ \lambda = g_{YM}^2 N_c = 2\pi g_s N_c, \tag{2.111a} \]

\[ \frac{1}{2} \frac{L^4}{\alpha'^2}. \tag{2.111b} \]

To obtain a meaningful theory, it is kept fixed in the large \( N_c \) limit, meaning \( g_s \to 0 \). Furthermore, we only know how to work on the field theory side in the weak coupling limit, meaning \( \lambda \ll 1 \). On the contrary, SUGRA is valid for weakly curved spacetime, which corresponds to a large gravitational radius \( L/\sqrt{\alpha'} \) and large \( \lambda \gg 1 \). On the one hand, this mismatch of regimes makes the duality hard to prove. On the other hand, this mismatch is the power of the duality. A strongly coupled field theory can be understood by understanding SUGRA in weakly curved space, which is used in the following.

Motivated by this, Maldacena conjectured that this duality is not only valid in the large \( N_c \) and low-energy limit. This generalizes the duality to

\[ \mathcal{N} = 4 \ SU(N_c) \text{ SYM theory on } \mathbb{R}^{3,1} \leftrightarrow \text{type IIB superstring theory on } \text{AdS}_5 \times S^5 \]

for general \( N_c \) and \( \lambda \). By Kaluza-Klein reduction, the gravity side can be reduced to a theory in five dimensions. Consequently, this duality is a realisation of the holographic principle. The fifth coordinate \( z \) has dimensions of length. Further analysis of the
degrees of freedom shows that $z \to 0$ corresponds to the UV and $z \to \infty$ to the IR. Especially, a UV cut-off in the field theory corresponds to the cut-off $\epsilon \leq z$ on the gravity side. This is further explained in appendix C. The mapping between parameters on the field theory and on the gravity side can be found in section A.2 of the appendix.

For the theories to be dual, the symmetries have to match. On the field theory side, the bosonic symmetries are the conformal symmetry $SO(4,2)$ and the R-symmetry $SU(4)_R \sim SO(6)$, which appears due to the $\mathcal{N} = 4$ supersymmetries and the corresponding indices $a = 1, \ldots, \mathcal{N}$. The first one agrees with the symmetry group of AdS$_5$.

In section 2.2.2, the Penrose-Brown-Henneaux transformation was introduced, which at the boundary explicitly reduces to a conformal transformation. For this reason, the field theory is said to live on the boundary of the space. The $SO(6)$ symmetry matches the symmetry of the compact space $S^5$.

**Operator mapping**

The above correspondence was made more precise in [12, 13]. There is a one-to-one map between operators, which transform in the same representation of the symmetry groups. On the field theory side, these operators are gauge-invariant, hence it has to be composite operators. To obtain fields transforming in an irreducible representation of $SU(4) \sim SO(6)$ on the gravity side, a Kaluza-Klein decomposition into $S^5$ spherical harmonics has to be performed. They are eigenmodes of the $S^5$ Laplace operator and give masses to their coefficients. For example, for a scalar field $\phi$ we obtain

\[
\phi(z, x, \Omega_5) = \sum_l \phi_l(z, x) Y_l(\Omega_5),
\]

\[
\Box_{S^5} Y_l(\Omega_5) = -\frac{1}{L^2} l(l + 4) Y_l(\Omega_5),
\]

\[
\Box_{AdS} \phi_l(z, x) = \frac{1}{L^2} l(l + 4) \phi_l(z, x) = m^2 \phi_l(z, x).
\]

The Laplace operator in AdS space and its near-boundary expansion are

\[
\Box_{AdS} \phi_l(z, x) = \frac{1}{L^2} \left( z^2 \partial_z - (d - 1) z \partial_z + z^2 \eta^{\mu\nu} \partial_\mu \partial_\nu \right) \phi_l(z, x),
\]

\[
\rightarrow \frac{1}{L^2} \left( z^2 \partial_z - (d - 1) z \partial_z \right) \phi_l(z, x) \quad \text{for } z \to 0.
\]

This equation is a second order differential equation and has in general two independent solutions. Asymptotically, the solutions have the form

\[
\phi(z, x) = \phi(0) z^{\Delta_-} + \phi(+) z^{\Delta_+}
\]

with

\[
\Delta_\pm = 2 \pm \sqrt{4 + m^2 L^2}.
\]

However, the two coefficients are not linearly independent. There is no solution to the Laplace equation in AdS (i.e. $\Box_{AdS} \phi_l = 0$), which is square-integrable. Therefore,
only one linear combination is smooth in the whole AdS-space. The boundary value of \( z^{-\Delta} \phi(z,x) \) determines the solution uniquely. In AdS space, a scalar field is dimensionless. Therefore, \( \phi(0) \) has dimension \( \Delta_+ = 4 - \Delta_- \) and \( \phi(+\) has dimension \( \Delta_+ \).

The dual operator is an operator \( O_\Delta \) with conformal dimension \( \Delta = \Delta_+ \). The dimensional analysis above suggest that \( \phi(0) \) acts as a source of this operator and \( \phi(+) \) is associated with the vacuum expectation value.

The precise form of the correspondence is

\[
Z_{\text{CFT}}[\phi(x)] = Z_{\text{AdS}}[\phi(z,x)], \tag{2.116a}
\]

\[
\langle \exp \left( \int \phi_0 O \right) \rangle_{\text{CFT}} = \exp (i S_{\text{AdS}}(\phi)). \tag{2.116b}
\]

\( Z_{\text{CFT}} \) is the partition function on the field theory side and \( Z_{\text{AdS}} \) is the partition function on the gravity side. The equivalence of both partition functions is called GKP-Witten relation. In the limit of weak curvature, the saddle-point approximation can be used on the gravity side. This relation can be used to calculate the connected \( n \)-point functions on the field theory side by differentiating with respect to \( \phi_0 \) and set \( \phi_0 = 0 \).

**Thermodynamics**

The near-horizon limit can be also be performed for non-extremal D3-branes [23], yielding

\[
ds^2 = \frac{r^2}{L^2} \left( -f(r)dt^2 + dx^idx^i \right) + \frac{L^2}{r^2} \left( dr^2 + r^2 d\Omega_5^2 \right), \tag{2.117a}
\]

\[
f(r) = 1 - \frac{r_h}{r^4}. \tag{2.117b}
\]

Performing the coordinate change \( z = L^2/r \) shows that this metric is the product of the AdS\(_5\)-Schwarzschild metric (2.50) and \( S^5 \). The dual field theory is \( \mathcal{N} = 4 \) SYM theory at temperature \( T = \frac{r_h}{2\pi L} = \frac{1}{\pi \rho} \). Applying the GKP-Witten relation (2.116a), the thermodynamics of the black hole describe the thermodynamics of the field theory. The entropy, the free energy and the average energy are in field theory quantities (c.f. equations (2.57) and (2.58))

\[
S = \frac{1}{2} \pi^2 N_c^2 T^3 \text{Vol}(\mathbb{R}^3), \tag{2.118a}
\]

\[
F = -\frac{1}{8} \pi^2 N_c^2 T^4 \text{Vol}(\mathbb{R}^3), \tag{2.118b}
\]

\[
\langle E \rangle = \frac{3}{8} \pi^2 N_c T^4 \text{Vol}(\mathbb{R}^3). \tag{2.118c}
\]
2.4.2 Adding flavour to AdS/CFT

The previously reviewed conjecture only contains adjoint degrees of freedom on the field theory side. However, it is well-known that QCD also has fundamental degrees of freedom. To obtain a theory close to QCD, it is therefore interesting to extend the AdS/CFT correspondence to field theories with flavour.

In [28], Karch and Katz developed a method to introduce fundamental flavour (i.e. fields which transform in the fundamental representation of $SU(N_c)$) in the AdS/CFT correspondence. The idea is to add additional $N_f$ probe $D^p$-branes, which extends the field theory to $\mathcal{N} = 4\ SU(N_c)$ SYM theory with $N_f$ matter multiplets. The complete action is

$$S = S_{\text{IIB}} + S_{\text{brane}}$$

and the solution differs from the known AdS$_5 \times $ S$^5$ solution of type IIB SUGRA. This is avoided when working in the probe approximation, where $N_f$ is kept fixed in the large $N_c$ approximation. The probe branes source a change of the bulk fields of order $N_f/N_c$, which is called backreaction of the probe branes on the bulk. In the large $N_c$ limit, this backreaction can be neglected. Therefore, the bulk fields are solved by the already known solution for $S_{\text{IIB}}$ and the brane fields are solved by minimizing $S_{\text{brane}}$ for the fixed bulk fields.

### Possible probe branes

There are three criteria which restrict the dimension of the probe branes $p$.

- To ensure stability, the $D^p$ brane has to be charged under a R-R sector gauge field and the intersection with the D3-branes has to be supersymmetric. Concretely, this restricts $p$ to odd numbers and the number of Neumann-Dirichlet directions (i.e. the number of directions in which only one of the branes extends) to a multiple of 4.

- The $D^p$-branes have to extend in the time and in the radial direction. Physically, this means that the fundamental degrees of freedom are not restricted at a specific energy scale or time.

- The $SU(N_f)$ symmetry should be global, which corresponds to a vanishing Yang-Mills coupling. According to equation (2.93), the Yang-Mills coupling is proportional to $\alpha'(p-3)/2$. Therefore, $p$ has to be larger than 3 to obtain the right behaviour in the limit $\alpha' \to 0$.

The $D^p$ probe branes which satisfy this constraints are shown in Table 2.4. Physical flavour propagate in all spacetime directions, requiring to use $D^7$-probe branes with the embedding which extends in the directions $(t, x^1, x^2, x^3, z, y^1, y^2, y^3)$.

On the gravity side, the DBI of the action for $N_f$ probe $D^7$-branes with $U(1)$ field strength is equation can be read off from equation (2.97) and yields

$$S_{\text{brane}} = -\tau_7 N_f \int d^8x \sqrt{-\det (g_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta})},$$

(2.120)
Table 2.4: Dp probe branes which satisfy constraints

<table>
<thead>
<tr>
<th>Nc D3</th>
<th>t</th>
<th>x1</th>
<th>x2</th>
<th>x3</th>
<th>z</th>
<th>y1</th>
<th>y2</th>
<th>y3</th>
<th>y4</th>
<th>y5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nf D7</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nf D7</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nf D5</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Nf D5</td>
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<td></td>
</tr>
</tbody>
</table>

where \( \tilde{\tau} = \tau_7/g_s \) is the tension of the branes, \( g_{\alpha\beta} \) is the induced metric on the brane and \( F \) is the field strength for the \( U(1) \) gauge field on the D7-brane. The D7-branes break the \( SO(6) \) symmetry of the compact space into a \( SO(2) \) symmetry for the directions transverse to the brane and a \( SO(4) \) symmetry for the directions parallel of the brane. By choosing appropriate coordinates for \( S^5 \), the metric can be written as

\[
L^2 d\Omega_5^2 = L^2 (d\theta^2 + \sin^2 \theta d\psi^2 + \cos^2 \theta d\Omega_3^2).
\]  
(2.121)

The \( SO(2) \) symmetry allows an embedding with \( \theta(z), \psi = 0 \). The induced metric on the brane is

\[
g_{\text{ind}} = \frac{L^2}{z^2} \left( \frac{dz^2}{b(z)} \left(1 + z^2 b(z) \theta'(z) \right) - b(z) dt^2 + d\vec{x}^2 \right) + L^2 \cos^2 \theta(z) d\Omega_3^2.
\]  
(2.122)

The D7-branes end at \( z_0 \), where the radius of \( S^3 \) shrinks to zero (i.e. \( \theta(z_0) = \pi/2 \)) or the time \( S^1 \) shrinks to zero (i.e. \( z_0 = z_h \)). Consequently, there are two qualitatively different situations. If the D7-branes end outside of the event horizon, the embedding is called Minkowski embedding. The geometry of the brane does not contain the black hole. This corresponds to a mass gap in the field theory, since the fundamental fields only exist above a specific energy scale and are absent in the IR. If however the D7 branes reach the event horizon, the embedding is called black hole embedding. The geometry of the branes contains the black hole. On the field theory side, this means that the temperature is high enough or equivalently the mass of the fundamental field is low enough to eliminate the mass gap. The critical embedding is the limiting case where the \( S^3 \) and the \( S^1 \) shrink to zero at the same time.

The embedding for vanishing worldvolume field strength was already calculated in [30] and [31]. On the field theory side, this corresponds to the absence of charge density, magnetic field and electric field. In this case, equation (2.120) simplifies to

\[
S_{\text{brane}} = -N_f \tilde{\tau}_7 \text{Vol}(S^3) L^5 \int d^5 x \frac{L^3}{z^5} \cos^3(\theta) \sqrt{1 + z^2 b\theta'^2}
\]

\[
= -\frac{t_0}{16\pi G} \int d^5 x \frac{L^3}{z^5} \cos^3(\theta) \sqrt{1 + z^2 b\theta'^2}.
\]  
(2.123)

Here, we defined \( t_0 \) as

\[
t_0 = 16\pi G N_f \tilde{\tau}_7 \text{Vol}(S^3) L^5 = \frac{\lambda N_f}{\pi^2 N_c}.
\]  
(2.124)
The equation of motion for the embedding function $\theta(z)$ is

$$3 \sin(\theta) \left(1 + z^2 b \theta'^2\right) + z \cos(\theta) \left((b - 4) \theta' - 2z^2 b(1 + b) \theta'' + zb \theta'''ight) = 0.$$  

(2.125)

To determine the initial conditions, it has to be distinguished between the two embeddings. For the Minkowski embeddings, the initial conditions are obtained by fixing the end of the brain $z_0$ and substituting $\theta(z_0) = \pi/2$ into equation (2.125). This yields

$$\theta(z_0) = \frac{\pi}{2},$$  

(2.126a)

$$\theta'(z_0) \to \infty.$$  

(2.126b)

For the black hole embeddings, evaluating equation (2.125) for $z = z_h$ yields

$$\theta_0 = \theta(z_h),$$  

(2.127a)

$$\theta'(z_h) = \frac{3}{4z_h} \tan \theta_0.$$  

(2.127b)

The constant solution is given by $\theta = 0$. Expanding the Lagrangian around this solution yields, after bringing the kinetic term to the appropriate form, a scalar field with mass squared $m^2 = -3/L^2$. This corresponds to a dual operator in the field theory with dimension $\Delta = 3$, which is a quark bilinear. The source is identified with the flavour mass $M_q$ and the vacuum expectation value with the analogue of the quark condensate $\langle \bar{\psi} \psi \rangle$.

Near the boundary $z = 0$, $\theta$ is given by

$$\theta = \theta'(0)z + \frac{1}{6} \theta''(0)z^3 + O(z)^5,$$

$$= \theta'(0) \frac{z}{\pi T} \frac{z_h}{z} + \frac{1}{6} \theta''(0) \frac{z^3}{\pi^3 T^3} \frac{z_h^3}{z_h^3} + O(z)^5.$$  

(2.128)

Following the conventions from [38][10], we work with the dimensionless quantities

$$\frac{m}{T} = \frac{\theta'(0)}{T},$$  

(2.129a)

$$\frac{M}{T} = \sqrt{2} \frac{\theta'(0)}{T},$$  

(2.129b)

$$c = \frac{\sqrt{2}}{3 \pi^3 T^3} \left(\theta''(0) - \theta'(0)^3\right).$$  

(2.129c)

The quark mass $M_q$ and the quark condensate $\langle \bar{\psi} \psi \rangle$ are

$$M_q = \frac{\sqrt{\lambda}}{2} \frac{M}{T},$$  

(2.130a)

$$\langle \bar{\psi} \psi \rangle = -\frac{1}{8} \sqrt{\lambda N_f N_c T^3} c.$$  

(2.130b)
The differential equation (2.125) can be solved numerically. Since there are two length scales (due to \( \bar{M} \) and \( T \)), the only relevant parameter is \( \bar{M}/T \). In the calculation, we can simply fix \( z_h = 1 \) and obtain result for a different temperature by using

\[
\theta(z) = \theta\big|_{z_h=1} \left( \frac{z}{z_h} \right),
\]

(2.131a)

\[
\theta(0) = \frac{1}{z_h} \theta\big|_{z_h=1}(0) = \pi T \theta\big|_{z_h=1}'(0).
\]

(2.131b)

Figure 2.4 shows the results for different embeddings. The result for the dependence between quark condensate and mass is shown in Figure 2.5. Near the critical solution, there is a range of \( \bar{M}/T \) where embeddings with different value for the quark condensate are possible. The physical solution is the one with the lowest free energy and there is a phase transition between the different embeddings. Later, the entanglement entropy is used to determine the free energy and to analyse this transition.

### 2.4.3 Holographic entanglement entropy

The calculation of the entanglement entropy in the field theory is rather complicated in the higher-dimensional case. Therefore, it is an important development in the AdS/CFT correspondence to be able to calculate the entanglement entropy holographically on the gravity side. The holographic entanglement entropy was conjectured by Ryu and Takayanagi [18,19]. A review can be found in [20].

The proposal is motivated by the area law for the entropy of a black hole [24–26] and the holographic entropy bound [3]. The space behind the horizon is not accessible. Since the entanglement entropy is the not accessible information about the complement of a region on the boundary of the AdS space, the idea is to hide part of the AdS space behind an imaginary horizon and calculate the corresponding entanglement entropy to

---

The same conventions (up to a missing factor of \( N_f \) for the quark condensate) were also used in [31].
obtain an entropy bound. By minimizing the corresponding horizon area, we find an upper bound for the entropy. Explicitly, the holographic entanglement entropy is

\[ S_{EE}(B) = \frac{A}{4G_N}, \]  

\[ A = \min_{\gamma_B |\partial\gamma_B = \partial B} \text{Area}(\gamma_B), \]  

where the surface \( \gamma_B \) is an equal time surface with the same boundary as the considered region \( B \). \( G_N \) is the Newton’s constant in five dimensions. Embedding the surface as \( x^\alpha_m(w^\alpha) \), \( \alpha = 1, 2, 3 \), the integral we have to minimize is

\[ A = \int d^3w \sqrt{\gamma}, \]  

\[ \gamma_{\alpha\beta} = g_{ab} \partial_\alpha x^a_m \partial_\beta x^b_m, \]  

where \( \gamma \) is the induced metric on the minimal surface.

Let us take a qualitative look at the minimizing surfaces, as shown in Figure 2.6. For zero temperature, the turning point of the minimal surface is deep in the bulk for a large region. It is clear that the holographic entropy for the complement is the same as for the original region, because they have the same boundary. This is expected for a pure state. For non-vanishing temperature, the minimizing surface only extends outside the horizon and approaches the horizon in the large-volume limit. An observer never has access to the inside of the black hole, therefore the black hole is always on the other side of the minimizing surface. The holographic entanglement entropy of the complement does no longer agree with the holographic entanglement of the region.
Since there are no exact solutions on the field theory side for general dimension, it is difficult to prove this conjecture. For $d = 2$, the holographic results agree with the field theory results already presented (see [18]).\footnote{For the massive case, the finite correlation length was used as a hard IR cut-off.} Therefore, the holographic entanglement entropy is not only a bound for the entropy but exactly the entropy in two dimensions. Furthermore, at non-vanishing temperature the minimizing surface for a large region spans the horizon and the dominant contribution to the holographic entanglement entropy is from the thermal entropy density. With this in mind, it is assumed that this dual construction agrees with the entanglement entropy in general dimension.

Additionally, the holographic entanglement entropy satisfies the subadditivity and strong subadditivity conditions, which obtain a geometrical meaning (e.g. for the strong subadditivity shown in Figure 2.7 and proven in [48, 49]). Further supporting evidence is the agreement with the area law for a UV cut-off. A UV cut-off in the field theory corresponds to a cut-off $\epsilon \leq z$ on the gravity side. Expanding a general $d + 1$-dimensional asymptotic AdS metric in the near-boundary limit, the holographic entanglement entropy produces the expected area law $S \sim 1/\epsilon^{d-2}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{strong_subadditivity}
\caption{Strong subadditivity of holographic entanglement entropy}
\end{figure}

Picture taken from [20], labeling and line-style modified
Chapter 3

Temperature contribution to the entanglement entropy

In this chapter, we study the entanglement of $\mathcal{N} = 4$ SYM theory at finite temperature. The calculation is performed by using Takayanagi’s and Ryu’s proposal for the holographic entanglement entropy as explained in section 2.4.3, where the entanglement entropy is calculated by

$$S_{EE}(B) = \frac{A}{4G_N}. \quad (3.1)$$

First, we review the result for vanishing temperature following [18, 19] and then I generalize this calculation to non-vanishing temperature.

In both cases, the entanglement entropy is calculated for a straight belt with width $l$ as shown in Figure 3.1. The belt is infinitely extended in the other spatial directions, which is regularized by the width $\tilde{l} \gg l$. In this limit, the minimal surface can be embedded independently of $x^2$ and $x^3$, which allows to parametrize the radial coordinate as $z(x^1)$. The advantage of choosing a belt is that the metric is independent of $x^1$. The corresponding conserved quantity of the minimal surface can be solved for $z'(x^1)$ instead of solving a second order differential equation. This is for example not possible for a circular disk. Furthermore, the entanglement entropy of a straight belt can be used to derive a measure for the effective degrees of freedom in form of the entropic c-function as introduced in [20].

3.1 Review of the zero temperature result

Let us first focus on the zero temperature case. The AdS metric is

$$ds_{AdS}^2 = \frac{L^2}{z^2} \left( dz^2 - dt^2 + dx^2 \right). \quad (3.2)$$
We minimize the area of a surface embedded as \((z(x^1), x^i)\). The area is (c.f. equation (2.133))

\[
A^{(0)}_{T=0} = l^2 \int_{-l/2}^{l/2} dx^1 \sqrt{g_{11}(z)g_{22}(z)g_{33}(z) \left( \left( \frac{dz}{dx^1} \right)^2 \frac{g_{zz}(z)}{g_{11}(z)} + 1 \right)},
\]

\[
= l^2 \int_{-l/2}^{l/2} dx^1 \frac{L^3}{z^3} \sqrt{1 + \left( \frac{dz}{dx^1} \right)^2}.
\]

(3.3)

In the following, the superscript \((0)\) is used to clarify that we mean quantities without flavour contribution. Since the metric is independent of \(x^1\), the quantity

\[
\frac{\delta \mathcal{L}_{\text{min}}}{\delta z'} z' - \mathcal{L}_{\text{min}} = \frac{L^3}{z^3} \sqrt{\left( \frac{dz}{dx^1} \right)^2} + 1
\]

(3.4)

is conserved along the minimal surface and the differential equation for the radial coordinate \(z\) is

\[
z' = \frac{\partial z}{\partial x^1} = \pm \frac{z^3}{z^3} \sqrt{1 - \frac{z^6}{\tilde{z}^6}},
\]

(3.5)

where \(\tilde{z}\) is the turning point of the surface. This allows to transform the integral in equation (3.3) in an integral over \(z\). Using the symmetry \(x^1 \to -x^1\), we choose the positive sign and have to add a factor of two when substituting \(dx^1\). The width \(l\) of the belt for the turning point \(\tilde{z}\) is

\[
l(\tilde{z}) = 2 \int_0^{\tilde{z}} dz \frac{dx^1}{dz} = \frac{2\sqrt{\pi} \Gamma \left( \frac{2}{3} \right)}{\Gamma \left( \frac{4}{3} \right)} \tilde{z}.
\]

(3.6)
3.2 Result for finite temperature

For finite temperature, there are two different candidates for the minimal surface: the minimal continuous one and a piece-wise continuous one, which consists of two pieces connecting the boundary with the horizon (i.e. $x^i = \text{const.}, \; i = 1, \ldots, 3$) and a piece along the horizon (i.e. $z = z_h$), as shown in Figure 3.2. By calculating the difference between the areas of this surfaces $\Delta A = A_c - A_d$, the authors of [37] showed numerically

![Figure 3.2: Different surfaces](image)

The red (dashed) surface is the piece-wise continuous one, the green (dotted) surface is the continuous one and the blue (dashed-dotted) region is the considered region $B$. The black circle indicates the horizon with $z = z_h$. After transforming the integral in equation (3.3), the minimal surface area is

\[
A^{(0)}_{\text{min}}(0) |_{T=0} = 2\tilde{l}^2 \int_\epsilon^z d\tilde{z} \frac{L^3}{\tilde{z}^3} \frac{1}{\sqrt{1 - \frac{\tilde{z}^6}{\epsilon^6}}},
\]

\[
= \frac{L^3\tilde{l}^2}{\epsilon^2} \frac{L^3\tilde{l}^2 \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right) \tilde{z}^{2}},
\]

\[
= \frac{L^3}{\epsilon^2} \text{Vol}(\partial B) - \frac{4\pi^{3/2} \Gamma\left(\frac{2}{3}\right)^3 L^3 \tilde{l}^2}{\Gamma\left(\frac{1}{6}\right)^3} \cdot \frac{1}{\tilde{l}^2},
\] (3.7)

where $\epsilon$ is the bulk cut-off $\epsilon \leq z$. The entanglement entropy in terms of field theory quantities (c.f. section A.2) is

\[
S_{EE}^{(0)} |_{T=0} = \frac{L^3\tilde{l}^2}{4G_N \epsilon^2} = \frac{L^3\tilde{l}^2 \Gamma\left(\frac{2}{3}\right)}{4G_N \Gamma\left(\frac{1}{6}\right) \tilde{z}^{2}},
\]

\[
= \frac{N_c^2}{2\pi \epsilon^2} \text{Vol}(\partial B) - \frac{2N_c^2 \pi^{1/2} \Gamma\left(\frac{2}{3}\right)^3 L^3 \tilde{l}^2}{\Gamma\left(\frac{1}{6}\right)^3} \cdot \tilde{l}^2.
\] (3.8)
that the continuous surface is always smaller and hence favourable. The area for the piece-wise continuous surface can easily be calculated analytically and is

\[ A_d^{(0)} = 2l^2 L^3 \int \frac{dz}{z} \frac{1}{\sqrt{b(z)}} + 4l^2 \frac{L^3}{z_h^2} = L^3 \pi^3 T^3. \]  

(3.9)

The associated entropy is

\[ \frac{A_d^{(0)}}{4G_N} = \frac{N_c^2 l^2}{2\pi^2} + \frac{N_c^2}{2} \text{Vol}(B) \pi^2 T^3. \]  

(3.10)

The first term is the cut-off dependent area term, which also appears for the continuous surface. The second term is the volume contribution due to the thermal entropy, which can be seen by comparing it with the entropy in equation (2.118a).

In the following, I derive the area of the continuous surface analytically. Both the width and the minimal surface area can be expressed as analytic functions of the turning point \( z_\star \). The integrands contain two different square-roots and can be written in terms of hypergeometric functions, yielding generalized hypergeometric functions as result. A short review of them and the solution of the occurring integrals are presented in section B of the appendix.

The metric is the AdS-Schwarzschild metric

\[ ds_{\text{AdS}}^2 = \frac{L^2}{z^2} \left( \frac{dz^2}{b(z)} - b(z) dt^2 + d\vec{x}^2 \right), \]  

(3.11a)

\[ b(z) = 1 - \frac{z^4}{z_h^4}, \]  

(3.11b)

where the horizon is at \( z_h = 1/\pi T \). The minimal surface area is

\[ A^{(0)} = l^2 \int_{-l/2}^{l/2} dx^1 \sqrt{\frac{g_{11}(z) g_{22}(z) g_{33}(z)}{b(z)}} \left( \left( \frac{dz}{dx^1} \right)^2 \frac{g_{zz}(z)}{g_{11}(z)} + 1 \right)^{1/2}, \]  

(3.12)

The conserved quantity of the minimal surface is

\[ \frac{\delta \mathcal{L}_{\text{min}}}{\delta z'} \frac{\partial z}{\partial x^1} - \mathcal{L}_{\text{min}} = \frac{L^3}{z^3} \left( \left( \frac{dz}{dx^1} \right)^2 + 1 \right)^{-1} = \frac{L^3}{z^3}, \]  

(3.13)

where \( z_\star \) is the turning point of the minimal surface. Solving this for \( \partial z/\partial x^1 \) yields

\[ \frac{\partial z}{\partial x^1} = \sqrt{b(z) \frac{z^3}{z^3_h}} \left( 1 - \frac{z^6}{z^6_\star} \right). \]  

(3.14)
3.2 Result for finite temperature

Like in the zero temperature calculation, the width is calculated by writing it as an integral over \( x^1 \) and transforming it into an integral over \( z \). The result is

\[
l(z_*) = 2 \int_0^{z_*} dz \frac{1}{z^2} = 2 \int_0^{z_*} dz \frac{1}{\sqrt{1 - \frac{z^4}{z_h^4}} \sqrt{1 - \frac{z^6}{z_h^6}}} \tag{3.15}\]

and can be solved using hypergeometric functions (c.f. solution for \( I_1(4,s) \) in section B.2.1 of the appendix)

\[
l(z_*) = \frac{2\sqrt{\pi} z_* \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{1}{6} \right)} {}_4F_3 \left( \begin{array}{c} \{ \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{5}{6} \} ; \{ \frac{7}{12}, \frac{2}{3}, \frac{13}{12} \} ; \frac{z_*^{12}}{z_h^{12}} \end{array} \right) + \frac{z_*^9}{6^9 z_h^9} {}_5F_4 \left( \begin{array}{c} \{ \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \frac{3}{2} \} ; \{ \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4} \} ; \frac{z_*^{12}}{z_h^{12}} \end{array} \right)
+ \frac{\sqrt{\pi} \Gamma \left( \frac{1}{3} \right) z_*^5}{15 \Gamma \left( \frac{2}{3} \right) z_h^5} {}_4F_3 \left( \begin{array}{c} \{ \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{7}{6} \} ; \{ \frac{11}{12}, \frac{4}{12}, \frac{17}{12} \} ; \frac{z_*^{12}}{z_h^{12}} \end{array} \right). \tag{3.16}\]

Later, it will only be possible to calculate the flavour contribution numerically. However, the thin belt limit (i.e. \( z_* \ll z_h \) and \( l \ll z_h \)) can still be solved analytically. To re-express this result in terms of the width instead of the turning point, I expand equation (3.16) for a small turning point and invert it

\[
l = \frac{2\sqrt{\pi} z_* \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{1}{6} \right)} + z_* O \left( \frac{z_*^4}{z_h^4} \right), \tag{3.17a}\]

\[
z_* = \frac{\Gamma \left( \frac{1}{6} \right) l}{2\sqrt{\pi} \Gamma \left( \frac{2}{3} \right) + l O \left( \frac{l^4}{z_h^4} \right)}. \tag{3.17b}\]

We see that in leading order, this agrees with the zero temperature result.

Since \( z_h \) is the only length scale, all solutions are equivalent. The width for the belt at general temperature can be calculated from the \( z_h = 1 \) result by

\[
l(z_*) = z_h \cdot l \big|_{z_h=1} \left( \frac{z_*}{z_h} \right). \tag{3.18}\]

Figure 3.3a shows the result.

Let us turn to the holographic entanglement entropy. The area of the surface with turning point \( z_* \) is

\[
A^{(0)} = 2L^3 l^2 \int_{z_*} z_* dz \frac{1}{\sqrt{1 - \frac{z^4}{z_h^4}} \sqrt{1 - \frac{z^6}{z_h^6}}} \tag{3.19}\]
and can be solved in terms of generalized hypergeometric functions (c.f. solution for $I_1(-2, s)$ in section B.2.1 of the appendix). The result is

$$A^{(0)} = \frac{L^3 \bar{p}^2}{\epsilon^2} + \frac{L^3 \bar{p}^2}{4} \frac{z^6}{z_h^8} F_4 \left( \left\{ 1, 2; \frac{1}{6}, 1, \frac{7}{6}, 2 \right\}; \left\{ 3, 5, 4, 5 \right\}; \frac{z^4}{z_h^{12}} \right)$$

$$+ \frac{L^3 \sqrt{\pi} \bar{p} \Gamma \left( \frac{4}{3} \right) z^2}{2 \Gamma \left( \frac{2}{3} \right) z_h^6} F_4 \left( \left\{ 1, 1, 5, 7 \right\}; \left\{ 5, 11, 4 \right\}; \frac{z^4}{z_h^{12}} \right)$$

$$- \frac{L^3 \sqrt{\pi} \bar{p} \Gamma \left( \frac{2}{3} \right) 1}{z^2} F_3 \left( \left\{ -1, 1, 1, 5 \right\}; \left\{ 1, 7, 2 \right\}; \frac{z^4}{z_h^{12}} \right). \quad (3.20)$$

The AdS/CFT dictionary yields

$$\frac{L^3}{4G_N} = \frac{N_c^2}{2\pi} \quad (3.21)$$

and the entanglement entropy is

$$S_{EE}^{(0)} = \frac{N_c^2 \bar{p}^2}{2\pi \epsilon^2} + \frac{N_c^2 \bar{p}^2}{8\pi} \frac{z^6}{z_h^8} F_4 \left( \left\{ 1, 2; \frac{1}{6}, 1, \frac{7}{6}, 2 \right\}; \left\{ 3, 5, 4, 5 \right\}; \frac{z^4}{z_h^{12}} \right)$$

$$+ \frac{N_c^2 \bar{p} \Gamma \left( \frac{4}{3} \right) z^2}{4 \sqrt{\pi} \Gamma \left( \frac{2}{3} \right) z_h^6} F_3 \left( \left\{ 1, 1, 5, 7 \right\}; \left\{ 5, 11, 4 \right\}; \frac{z^4}{z_h^{12}} \right)$$

$$- \frac{N_c^2 \bar{p} \Gamma \left( \frac{2}{3} \right) 1}{2 \sqrt{\pi} \Gamma \left( \frac{1}{6} \right) z^2} F_3 \left( \left\{ -1, 1, 1, 5 \right\}; \left\{ 1, 7, 2 \right\}; \frac{z^4}{z_h^{12}} \right). \quad (3.22)$$

For a thin belt, this reproduces the zero temperature result

$$S_{EE}^{(0)} = \frac{N_c^2 \bar{p}^2}{2\pi \epsilon^2} - \frac{N_c^2 \bar{p} \Gamma \left( \frac{2}{3} \right) \frac{1}{z^2}}{2 \sqrt{\pi} \Gamma \left( \frac{1}{6} \right) z_h^6} F_3 \left( \left\{ -1, 1, 1, 5 \right\}; \left\{ 1, 7, 2 \right\}; \frac{z^4}{z_h^{12}} \right)$$

$$= \frac{N_c^2 \bar{p}^2}{2\pi \epsilon^2} - \frac{2N_c^2 \pi^{1/2} \Gamma \left( \frac{3}{2} \right)^3 \bar{p}^2}{\Gamma \left( \frac{1}{6} \right)^3} \frac{1}{l^2} + \frac{1}{l^2} F_3 \left( \left\{ -1, 1, 1, 5 \right\}; \left\{ 1, 7, 2 \right\}; \frac{z^4}{z_h^{12}} \right). \quad (3.23)$$

The entropy can be written as a function of the dimensionless quantities $z_*/z_h$ and $\epsilon/z_h$

$$S_{EE}(z_*, \epsilon) = \frac{1}{z_h} S_{EE}|_{z_h=1} \left( \frac{z_*}{z_h}, \frac{\epsilon}{z_h} \right). \quad (3.24)$$

Figure 3.3 shows the width of the belt and the cut-off independent part of the entanglement entropy, i.e. we subtract the $1/\epsilon^2$ term. The plots compare the finite temperature results to the zero temperature results, for which $z_h = 1/\pi T$ is a finite constant. The most noticeable difference is the linear term in the entanglement entropy for non-vanishing temperature, which is the expected volume term for the thermal entropy. In the following, this volume term is analysed separately, before subtracting it to obtain the ‘corrected’ entanglement entropy.
3.2 Result for finite temperature

The entanglement entropy is plotted in units of $\tilde{l}^2 T^2 N_c^2$.

3.2.1 Thermodynamics of $\mathcal{N} = 4$ SYM theory

In this section we separate the volume term, which arises due to the thermal entropy. To do that, we consider the large-volume limit (i.e. $z_+ \to z_h$, $l \to \infty$), where this term dominates. In this limit, the leading order contribution to the width and to the entanglement entropy are coming from the divergent part of the integrands. In the following, subleading contributions with respect to this limit are neglected. This also includes the cut-off dependent term, which is independent of $l$. The entanglement entropy in this limit is

$$S_{EE}^{(0)} = \frac{L^3 \tilde{l}^2}{z_h^2 2G_N} \int_0^{z_+} ds \frac{1}{\sqrt{1 - s^4} \sqrt{1 - s^6 \frac{z_+}{z_h}} s^{-3}},$$

and the width $l$ is

$$l = 2z_h \int_0^{z_+} ds \frac{1}{\sqrt{1 - s^4} \sqrt{1 - s^6 \frac{z_+}{z_h}} s^{3 \frac{z_+}{z_h}^2}};$$

$$(3.26)$$

The subleading contributions are finite in this limit. The divergent parts of both integrals are proportional, which yields a volume term. The entanglement entropy
density coincides with the thermal entropy (c.f. equation 2.118a)
\[ S_{EE}^{(0)} = \frac{L^3}{4G_N} - \pi^3 \text{Vol}(B)T^3 + \cdots = \frac{N_c^2\pi^2}{2} \text{Vol}(B)T^3 + \cdots, \] (3.27a)
\[ = S \frac{\text{Vol}(B)}{\text{Vol}(\mathbb{R}^3)} + \cdots. \] (3.27b)

Let us reproduce the free energy and average energy from my result. Since both are proportional to the volume \( \text{Vol}(\mathbb{R}^3) \), we calculate the densities, but use the same symbols as introduced above. Using equations (2.33), the free energy and the average energy can be calculated from the entropy, which yield
\[ F_{\text{adj.}} = -\int dT \frac{S}{\text{Vol}(\mathbb{R}^3)}, \]
\[ = -\frac{N_c^2\pi^2}{8} T^4, \] (3.28a)
\[ \langle E \rangle = F + TS/\text{Vol}(\mathbb{R}^3), \]
\[ = \frac{3N_c^2\pi^2}{8} T^4 \] (3.28b)
and agree with the known results in equation (2.118). The stress energy tensor in the field theory is
\[ \langle T_{\mu\nu} \rangle = \left( \begin{array}{cc} \langle E \rangle & -F \\ -F & -F \end{array} \right) = \frac{3N_c^2\pi^2}{8} T^4 \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \] (3.29)
and is traceless.

### 3.2.2 ‘Corrected’ entanglement entropy

Now that we identified the thermal contribution, we can deduct it from the entanglement entropy. I define the ‘corrected’ entanglement entropy as
\[ \tilde{S}_{EE}^{(0)} = S_{EE}^{(0)} - \frac{N_c^2\pi^2}{2} \tilde{l}^2 T^3. \] (3.30)

Especially interesting is the large-volume limit, i.e. \( l \to \infty \). Performing this limit with our known results can lead to inaccuracies, because it contains two divergent parts which cancel each other. Fortunately, the subleading contribution to the entanglement entropy is finite. Hence, equation (3.30) can be rewritten as an integral, for which the limit \( l = \infty \) (i.e. \( z_* = z_h \)) is well defined and can be calculated (c.f. \( I_3 \) in section B.2.3 of the appendix). The result is
\[ \tilde{S}_{EE}^{(0)}(z_* = z_h) = \pi N_c^2 T^2 \tilde{l}^2 \int_{\epsilon/z_h}^{1} ds \frac{\sqrt{1 - s^6}}{s^3\sqrt{1 - s^4}}, \]
\[ = \frac{N_c^2}{2\pi\epsilon^2} \text{Vol}(\partial B) - 1.04598 N_c^2 T^2 \tilde{l}^2. \] (3.31)
3.3 Discussion

Let us first return to the zero temperature result. Due to the shape of the region, the entanglement entropy is proportional to $\tilde{l}^2$ and the width is independent of $\tilde{l}$. Since the only other length scale is the turning point, all terms of the width and the entanglement entropy are fixed up to proportionality constants by dimensional analysis. This implies for the turning point that it is proportional to the width of the belt. The physical interpretation of this is that the entanglement entropy for a wide belt is also influenced by low energy physics, whereas the entanglement for a thin belt only captures high energy physics. Besides the overall factor of $\tilde{l}^2$, the entanglement entropy is proportional to $N_c^2$, i.e. it is proportional to the number of degrees of freedom. This is expected for the entanglement entropy in an even dimensional CFT. The cut-off dependent term of the entropy is the expected area law for the cut-off, as explained in section 2.1.2. The cut-off dependent part is state-independent [50], which implies that it agrees with the finite temperature result.

For non-vanishing temperature, the minimal surfaces end outside of the event horizon and approach it in the large-volume limit. The system has an additional length...
scale due to the temperature. This yields a dimensionless quantity: the ratio between turning point and horizon position $z_\star/z_h$. For finite temperature, dimensional analysis only yields $l \propto z_\star \cdot \omega_l(z_\star/z_h)$, where $\omega_l$ is an unknown function, and the analogous result for the entanglement entropy, which makes the result less predictable. For the entanglement entropy, dimensional analysis would allow further cut-off dependent terms, because $\tilde{l}/z_h$ is dimensionless. However, this cut-off dependent terms are due to UV divergences and are therefore temperature independent.

The entanglement entropy approaches the thermal entropy in the large-volume limit. On the gravity side, this volume term arises because the continuous surface is wrapping part of the horizon, yielding a large contribution from the horizon area. The thereby obtained thermal entropy is identical to known results (see [51, 52] or equations (2.58)). Dimensional analysis fixes this result already up to a proportionality constant. Furthermore, the stress energy tensor is automatically traceless, as expected in a conformal theory.

The ‘corrected’ entanglement entropy $\tilde{S}_{EE}$ is the difference between the entanglement entropy and the volume term associated with the thermal entropy. My result agrees with the numerical result in [37], as shown in Figure 3.5. Let us compare this result to the result for vanishing temperature. Figure 3.6a shows the cut-off independent part of the ‘corrected’ entanglement entropy for vanishing and non-vanishing temperature. The difference between them

$$
\Delta \tilde{S}_{EE}^{(0)} = \tilde{S}_{EE}^{(0)}|_{T=0} - \tilde{S}_{EE}^{(0)}$

is shown in Figure 3.6b. Both results agree for $l = 0$. The difference is monotonically increasing with the width $l$ and approaches asymptotically a constant. This constant is due to the finite asymptotic value of the entanglement entropy at non-vanishing temperature (see equation (3.31)). Physically, this means that the system is less entangled

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There is an additional factor of two in this result, which is likely due to different conventions.
3.3 Discussion

Figure 3.6: Comparison of entanglement entropy to zero temperature result

The entanglement entropy is plotted in units of $\tilde{l}^2 T^2 N_c^2$. At finite temperature, for increasing temperature, the difference grows.\footnote{When plotting the difference $\Delta S_{EE}^{(0)}$ over $l$, a higher (lower) temperature means a compression (stretching) in the $x$-direction and a stretching (compression) in the $y$-direction. This increases (decreases) the difference for a fixed width.} This means the hotter the plasma is, the faster long range correlations get washed out. In the dual description, the finite temperature corresponds to a black hole, which changes the IR behaviour of the metric. For a larger region, the turning point lies closer to the horizon and the area is more influenced by the black hole. At the same time, increasing the temperature shifts the horizon closer to the boundary and the entanglement entropy for smaller regions is more influenced by the metric deformation.

An important feature is the finite asymptotic value (for $l \to \infty$) of the ‘corrected’ entanglement entropy. Asymptotically the entanglement entropy follows an area law, which indicates that the theory has a finite correlation length. This is a well-known phenomenon for a plasma at high temperature. At long distance, a test-charge is screened by the plasma and the potential decreases exponentially. Therefore, the temperature induces a finite correlation length and an effective mass for the massless particles. This phenomenon is called Debye screening. For strongly coupled $\mathcal{N} = 4$ SYM theory, it was first found in [23] and further investigated in [53–55] by considering holographic Wilson loops.

Another way to understand the physical interpretation of the entanglement entropy is to look at the entropic c-function in Figure 3.4. We can nicely see how the in equation (2.26) defined function satisfies the criteria for a c-function. Firstly, the entropic c-function is non-increasing along the RG flow. Secondly, the entropic c-function is stationary at the UV and the IR fixed point, which correspond to $l = 0$ and $l = \infty$ respectively. For zero temperature, the c-function is constant, which is expected for a conformal theory. Furthermore, the entropic c-function is proportional to $N_c^2$, which is proportional to the central charge of the theory. For non-vanishing temperature, the
temperature can be neglected in the high energy limit and the theory has the same UV fixed point, which is confirmed by my results. At the IR fixed point, the finite temperature theory seems to approach a UV fixed point with vanishing central charge and no degrees of freedom. The physical reason for this is that the temperature induces a mass gap, the fields become heavy and get integrated out.

Summarizing this chapter about entanglement entropy in \( \mathcal{N} = 4 \) SYM theory, my calculations both reproduced the known results for the thermal entropy as well as the numerical result for the ‘corrected’ entanglement entropy. Furthermore, I found an additional area law, which indicates a mass gap and a finite correlation length in the field theory. The asymptotic value is proportional to the temperature squared and caused by Debye screening. Additionally, I analysed the generalization of the entropic c-function by [20]. For finite temperature, it satisfies the criteria for a c-function and gives a meaningful interpretation as effective degrees of freedom. For \( \mathcal{N} = 4 \) SYM theory at finite temperature, we saw that it has the same UV fixed point as the zero temperature theory, but there are no degrees of freedom in the IR, because all fields are heavy due to the induced mass and are integrated out.
Chapter 4

Flavour contribution to the entanglement entropy

In the previous chapter, we examined the entanglement entropy of the adjoint degrees of freedom in $\mathcal{N} = 4$ SYM theory at finite temperature. This chapter analyses the leading order flavour correction to entanglement entropy. The flavour degrees of freedom are obtained by adding $N_f$ probe D7-branes as explained in section 2.4.2. If the ratio $\lambda N_f/N_c$ is small, the probe branes are only a small perturbation of the background. This makes it possible to treat $\lambda N_f/N_c$ as expansion parameter and calculate leading order corrections to the bulk fields. For the entanglement entropy, a method to calculate the leading order correction was developed by Chang and Karch [33]. In the following, we review this method for D7-branes, before applying it to probe branes with vanishing worldvolume gauge field.

4.1 Method

The method by Karch and Chang is based on Ryu’s and Takayanagi’s proposal for the holographic entanglement entropy, which is

$$S_{EE} = \frac{1}{4G_N} \int d^3w \sqrt{\gamma},$$

(4.1)

where $\gamma$ is the induced metric on the minimal surface. For a finite number of flavour $N_f$, the backreaction on the bulk metric vanishes in the large $N_c$ limit. Flavour corrections only become finite in the large $N_f$ limit. The appropriate inequalities, which make it possible to expand the backreaction order by order, are

$$\frac{L^3}{G_N} \gg \frac{t_0 L^3}{16\pi G_N} \gg 1,$$

(4.2a)

$$\frac{2N_c^2}{\pi} \gg \frac{\lambda N_f N_c}{8\pi^4} \gg 1.$$  

(4.2b)

This implies that the number of flavour is small compared to the number of adjoint degrees of freedom. This inequalities also insure that both the bulk action and the brane action can be treated classically.
To calculate the leading order flavour correction to the entanglement entropy, we have to calculate the change due to a small metric perturbation. The entanglement entropy depends on the metric in two ways: explicitly through the induced metric on the minimal surface and implicitly through the embedding of the minimal surface $x_m^a(x^µ)$, $a = 1, \ldots , 8$ (c.f. equation (2.133)). Since the embedding minimizes the area, the correction due to the changed embedding is of higher order and can be neglected. Therefore, the leading order flavour correction to the entanglement entropy is

$$
\delta S_{EE} = \frac{1}{4G_N} \int d^3w \sqrt{\gamma} \left( \frac{1}{2} T_{ab}^{\text{min}} (\delta g)_{ab} + \frac{\delta \sqrt{\gamma}}{\delta x_m^a} \delta x_m^a \right),
$$

$$
= \frac{1}{8G_N} \int d^3w \sqrt{\gamma} T_{ab}^{\text{min}} (\delta g)_{ab}, \quad (4.3)
$$

where $T_{ab}^{\text{min}}$ is the ‘stress energy tensor’ of the minimal surface

$$
T_{ab}^{\text{min}} = \frac{2}{\sqrt{-g_{\text{brane}}}} \delta g_{ab} \left|_{x^a \to x_m^a(w^a), \ g \to g(0)} \right.. \quad (4.4)
$$

Since the metric perturbation is already of first order, the ‘stress energy tensor’ is evaluated for the background metric. In particular, this means that the embedding of the probe branes has to be solved only in the probe approximation.

The metric perturbation $\delta g$ is sourced by the stress energy tensor of the probe brane. For a general probe D7-branes action

$$
S_{\text{brane}} = \int d^8z L_{\text{brane}} \quad (4.5)
$$

the stress energy tensor is

$$
T_{\text{brane}}^{ab} = -\frac{2}{\sqrt{-g_{\text{brane}}}} \frac{\delta L_{\text{brane}}}{\delta g_{ab}} \left|_{x^a \to x_p^a(z^i)} \right., \quad (4.6)
$$

where $x_p$ is the embedding and $g_{\text{brane}}$ is the induced metric on the probe branes. The metric perturbation is

$$
(\delta g)_{ab} = 8\pi G_N \int d^8z \sqrt{-g_{\text{brane}}(x_p(z))} \ G_{abcd}(x, x_p(z)) \ T_{\text{brane}}^{cd}(z), \quad (4.7)
$$

where $G_{abcd}$ is the Green’s function of ten dimensional linearised Einstein gravity. The leading order flavour correction to the entanglement entropy can be written as a double integral

$$
\delta S_{EE} = \pi \int d^3w \sqrt{\gamma} \int d^8z \sqrt{-g_{\text{brane}}(w)} T_{ab}^{\text{min}}(w) G_{abcd}(x_m(w), x_p(z)) T_{\text{brane}}^{cd}(z). \quad (4.8)
$$

In general, the probe branes source a flavour correction to the complete ten dimensional metric. However, we are only interested in metric perturbations which couple to
$T_{\text{min}}$. The minimal surface is a codimension-2 surface, which fills the complete compact space $S^5$. The internal components of the trace-reversed tensor of $T_{\text{min}}$

$$\tilde{T}^{ab}_{\text{min}} = T^{ab}_{\text{min}} - \frac{1}{8} g^{ab} T^{c}_{\text{min} c}$$  \hspace{1cm} (4.9)

vanish and the metric perturbation in this directions has nothing to couple to (c.f. equation (2.38)). The relevant metric perturbations are therefore sourced by the scalar spherical harmonics of the AdS part of $T_{\text{brane}}$. Hence, the compact space can simply be integrated out and we may work with the effective energy momentum tensor.\footnote{This is not explicitly written out before, because the internal integral is already integrated out. This is the reason that the holographic entanglement entropy contains the five-dimensional Newton’s constant.} Concretely, this means we work with the effective actions

$$S_{\text{bulk}} = \frac{1}{16\pi G_N} \int d^5 x \sqrt{-g_{\text{AdS}}} \left( R[g_{\text{AdS}}] + \frac{12}{L^2} \right),$$

$$S_{\text{brane}} = \int d^5 x \mathcal{L}_{\text{brane}},$$ \hspace{1cm} (4.10a) \hspace{1cm} (4.10b)

where the brane Lagrangian $\mathcal{L}_{\text{brane}}$ is

$$\mathcal{L}_{\text{brane}} = -\frac{t_0}{16\pi G_N \text{Vol}(S^3) L^5} \int d^3 x_i \sqrt{-\text{det} \left( g_{\text{ind}} + 2\pi \alpha' F \right)},$$ \hspace{1cm} (4.11)

The (effective) stress energy tensor is

$$T^{MN}_{\text{brane}} = \frac{-2}{\sqrt{-g_{\text{AdS}}} \delta g^{MN}} \frac{\delta \mathcal{L}_{\text{brane}}}{\delta \delta g^{MN}} \bigg|_{x^i \rightarrow x_i^p(x')}. \hspace{1cm} (4.12)$$

When following the formal approach to calculating the metric perturbation, the Green’s function has to be known completely. It is more practical to choose a suitable ansatz for the metric perturbation $\delta g$, expand the Einstein equations in leading order in $t_0$ and solve the resulting differential equation for the metric perturbation. This bypasses the calculation of the metric perturbation in equation (4.7) and we can calculate the flavour correction to the entanglement entropy by calculating equation (4.3) instead of the double integral.

However, there are restrictions to this ansatz. In equation (4.7), the Green’s function for linearised gravity is used. However, the probe branes also source other fields than the metric, which appear at quadratic or higher order in the Einstein-frame stress tensor. Consequently, as long as a bulk field is zero in the unperturbed geometry, the corresponding terms in the stress tensor are of higher order of $t_0$ and negligible. For the AdS$_5 \times S^5$ solution only $\mathcal{C}_{(4)}$ is non-vanishing (c.f. equation 2.99d). As long as $F \wedge F$ vanishes, the four-form is not sourced by the probe branes and the Green’s function for linearised gravity can be used.\footnote{This is qualitatively explained in [33]. A more detailed calculation using Kaluza Klein decomposition can be found in [35].}
There is one subtlety we have to consider. The backreaction of the metric also changes the boundary metric. Therefore, we potentially have to rescale the spacetime coordinates to obtain the Minkowski metric in the field theory. This gives an additional correction $\delta_2 S_{EE}$ to the entanglement entropy. The complete flavour correction to the entanglement entropy is

$$\delta S_{EE} = \delta_1 S_{EE} + \delta_2 S_{EE},$$

$$\delta_1 S_{EE} = \frac{1}{8G_N} \int d^3w \sqrt{\gamma} T^{ab}_{\text{min}} (\delta g)_{ab}. \tag{4.13b}$$

### 4.2 Backreaction of the metric

The first step in applying the aforementioned method is to calculate the leading order backreaction on the metric. In the following, we consider probe D7-branes without worldvolume gauge field (i.e. $F = 0$). They satisfy the condition $F \wedge F = 0$ trivially. The ansatz for the metric perturbation is

$$g_{\text{AdS}} = \frac{L^2}{z^2} \left( \frac{dz^2}{b} - b dt^2 + d\vec{x}^2 \right) + t_0 \delta g_{\text{AdS}}, \tag{4.14a}$$

$$\delta g_{\text{AdS}} = \frac{L^2}{z^2} \left( f(z) \frac{dz^2}{b} - h(z) b dt^2 + j(z) d\vec{x}^2 \right), \tag{4.14b}$$

which is the generalization of the ansatzes used in [34] and [36].

Linearising the Einstein equations (2.36a) yields

$$G^{zz} + \Lambda g^{zz} = \frac{3t_0 z^2 b(z)}{L^4} \left[ z (b(z) (h'(z) + j'(z)) + 2j'(z)) + 4f(z) \right] + O(t_0)^2, \tag{4.15a}$$

$$G^{tt} + \Lambda g^{tt} = -\frac{3t_0 z^2}{L^4 b(z)} \left[ 2z j'(z) - z b(z) (f'(z) + zj''(z) - j'(z)) \right] + 4f(z) + O(t_0)^2, \tag{4.15b}$$

$$G^{ij} + \Lambda g^{ij} = -\frac{t_0 z^2 \delta_{ij}}{L^4} \left[ -12f(z) + z(b(z) + 2)f'(z) + 3z(b(z) - 2)h'(z) \right. \right. $$

$$\left. + z^2 b(z) h''(z) + 2z^2 b(z) j''(z) + 2z(b(z) - 4)j'(z) \right] + O(t_0)^2. \tag{4.15c}$$

Although we have three differential equations, these equations are not independent and we cannot determine all three functions $f$, $h$, and $j$. Equation (4.15a) can be used to express $f$ in terms of $j$ and $h$. Using furthermore the equation of motion for $\theta$, the equations (4.15b) and (4.15c) simplify to one differential equation. The reason for this is the possible gauge transformation $z \rightarrow z + t_0 \xi(z)$. This transformation does not change the form of our ansatz, but transforms the functions $f$, $j$, and $h$. Instead of calculating these three functions, it is necessary to solve the differential equations for the two gauge-invariant combinations

$$\kappa_1(z) = b(z) f(z) + zb(z) j'(z) - 2j(z)(b(z) - 1), \tag{4.16a}$$

$$\kappa_2(z) = b(z) h(z) + (b(z) - 2) j(z). \tag{4.16b}$$
The Einstein tensor in terms of the gauge-invariant combinations is
\[ G^{tt} + \Lambda g^{tt} = -\frac{3t_0 z^7}{L^4 b(z)} \left( \frac{\kappa_1(z)}{z^4} \right)', \]
(4.17a)
\[ G^{zz} + \Lambda g^{zz} = \frac{3t_0 z^7}{4L^4} \left( \frac{\kappa_2'(z)}{z^3} \right)' + \frac{z \partial_z G^{zz}(z) + z \Lambda \partial_z g^{tt} - 6G^{zz}(z) - 6\Lambda g^{zz}}{4b(z)}. \]
(4.17b)

For vanishing worldvolume field strength, the brane action in equation (4.11) simplifies to
\[ L_{\text{brane}} = -\frac{t_0}{16\pi G_N} \frac{1}{\text{Vol}(S^3)} L^5 \int d^3 x_i \sqrt{-\det g_{\text{ind}}}, \]
\[ = -\frac{t_0}{16\pi G_N} \frac{1}{L^2} \cos^3 \theta \sqrt{-g_{zz}g_{tt}g_{11}g_{22}g_{33}} (1 + \theta'(z)^2 g_{\theta\theta} g^{zz}). \]
(4.18)

The stress energy tensor for this action is in leading order
\[ T_{\text{brane}}^{zz} = \frac{1}{8\pi G_N} (G^{zz} + \Lambda g^{zz}) = \frac{t_0 z^2 b(z)}{16\pi G_N L^4} \frac{\cos^3(\theta(z))}{\sqrt{z^2 b(z) \theta'(z)^2 + 1}}, \]
(4.19a)
\[ T_{\text{brane}}^{tt} = \frac{1}{8\pi G_N} (G^{tt} + \Lambda g^{tt}) = -\frac{t_0 z^2}{16\pi G_N L^4 b(z)} \frac{\cos^3(\theta(z))}{\sqrt{z^2 b(z) \theta'(z)^2 + 1}}, \]
(4.19b)
\[ T_{\text{brane}}^{ij} = \frac{1}{8\pi G_N} (G^{ij} + \Lambda g^{ij}) = \frac{t_0 z^2}{16\pi G_N L^4} \frac{\cos^3(\theta(z))}{\sqrt{z^2 b(z) \theta'(z)^2 + 1}}. \]
(4.19c)

Combining this with equation (4.17) yields the differential equations
\[ \left( \frac{\kappa_1(z)}{z^4} \right)' = -\frac{\cos^3(\theta(z))}{3z^5} \sqrt{z^2 b(z) \theta'(z)^2 + 1}, \]
(4.20a)
\[ \left( \frac{\kappa_2(z)}{z^3} \right)' = \frac{2(1 - b) \theta^2 \cos^3(\theta(z))}{3z^3 \sqrt{z^2 b(z) \theta'(z)^2 + 1}} \geq 0. \]
(4.20b)

This equations already fix a part of the boundary expansion of the gauge-invariant combinations, which is for \( \kappa_1 \)
\[ \kappa_1(0) = f(0) = \frac{1}{12}, \]
(4.21a)
\[ \kappa_1'(0) = f'(0) + j'(0) = 0, \]
(4.21b)
\[ \kappa_1''(0) = f''(0) + 2j''(0) = -\frac{\theta'(0)^2}{3}, \]
(4.21c)
\[ \kappa_1^{(3)}(0) = f^{(3)}(0) + 3j^{(3)}(0) = 0 \]
(4.21d)
and for \( \kappa_2 \)
\[ \kappa_2'(0) = h'(0) - j'(0) = 0, \]
(4.22a)
\[ \kappa_2''(0) = h''(0) - j''(0) = 0, \]
(4.22b)
\[ \kappa_2^{(3)}(0) = h^{(3)}(0) - j^{(3)}(0) = 0. \]
(4.22c)
\( \kappa_2(0), \kappa_1^{(4)}(0) \) and \( \kappa_2^{(4)}(0) \) are not fixed by this expansions and have to be either fixed as initial conditions for the differential equations or read of from the complete solution.

We will choose the second alternative and fix the initial conditions at the end of the brane. For the Minkowski embedding, the D7-branes end at \( z_0 < z_h \). At this point, the perturbed metric is joined with the unperturbed one, hence Israel junction conditions have to be satisfied \([56, 57]\). This means that the \( \mathcal{O}(t_0) \) contribution to the induced metric \( \gamma_{z=z_0} \) and to the induced extrinsic curvature \( K \) on the hypersurface \( z = z_0 \) vanish to avoid curvature singularities. The induced metric is

\[
\gamma_{z=z_0} = \frac{L^2}{z^2} \left( -(1 + t_0 h(z_0)) b(z_0) dt^2 + (1 + t_0 j(z_0)) dx^2 \right).
\]  

(4.23)

Matching this to the unperturbed metric at \( z_0 \) yields \( h(z_0) = j(z_0) = 0 \). Furthermore, the leading order correction to the induced extrinsic curvature \( K \) is

\[
K_{ab} = n^c \gamma_{ab,c} + n^c \gamma_{cb} + n^c \gamma_{ca} = K^{(0)}_{ab} + t_0 K^{(1)}_{ab},
\]

(4.24a)

\[
K^{(1)}_{zz} = 0,
\]

(4.24b)

\[
K^{(1)}_{tt} = \frac{L \sqrt{b(z_0)}}{z_0^2} \left( z_0 b(z_0) h'(z_0) - (b(z_0) - 2) f(z) \right),
\]

(4.24c)

\[
K^{(1)}_{ij} = - \delta_{ij} \frac{L \sqrt{b(z_0)}}{z_0^2} \left( f(z) + z j'(z_0) \right).
\]

(4.24d)

Therefore, the initial conditions are

\[
f(z_0) = z_0 \frac{b(z_0)}{b(z_0) - 2} h'(z_0),
\]

(4.25a)

\[
f(z_0) = -z_0 j'(z_0)
\]

(4.25b)

or in terms of the gauge-invariant combinations

\[
\kappa_1(z_0) = 0, \quad \kappa_2(z_0) = 0,
\]

(4.26a)

\[
\kappa_1'(z_0) = 0, \quad \kappa_2'(z_0) = 0.
\]

(4.26b)

For the black hole embeddings, the D7-branes end at \( z_0 = z_h \). It is difficult to obtain junction conditions, because our coordinates are singular at the horizon. Therefore, we choose initial conditions for which the metric perturbation has the same physical effects as for the Minkowski embeddings. When the metric ends at \( z_0 < z_h \), the metric at the horizon is not changed. Especially, this means that the temperature and the entropy (i.e. the horizon area) stay the same. We take this as restrictions for the black hole embeddings. The change of the horizon area is proportional to \( j(z_h) \), yielding \( j(z_h) = 0 \) and automatically \( \kappa_1(z_h) = \kappa_2(z_h) = 0 \). To calculate the temperature change due to the flavour branes, we have to expand the metric \((4.14a)\) around \( z_h \). The temperature of the perturbed black hole is

\[
T = \frac{1}{\pi z_h} \left( 1 + t_0 \frac{h(z_h) - f(z_h))}{2} \right).
\]

(4.27)
Demanding that the leading order corrections vanishes yields \( h(z_h) = f(z_h) \) and \( \kappa'_1(z_h) = \kappa'_2(z_h) \). \( \kappa'_2(z_h) \) is obtained by evaluating equation (4.20a) at the horizon. The complete initial conditions for black hole embeddings are therefore

\[
\begin{align*}
\kappa_1(z_h) &= 0, \\
\kappa_2(z_h) &= 0, \\
\kappa'_2(z_h) &= \kappa'_1(z_h) = -\frac{\cos^3(\theta(z_h))}{3z_h}.
\end{align*}
\]

### 4.2.1 Review of the zero temperature results

Before calculating the backreaction at finite temperature, we first check my results by rederiving the leading order backreaction at zero temperature and compare it to the results in [34]. The zero temperature case corresponds to the limit \( z_h \to \infty \) and \( b(z) = 1 \). The embedding is known analytically and is in our coordinates

\[
\begin{align*}
\cos^2 \theta &= 1 - m^2 z^2, \\
\theta'(0) &= m,
\end{align*}
\]

where the probe branes end at \( z_0 = 1/m \). The gauge-invariant combinations \( \kappa_1 \) and \( \kappa_2 \) are

\[
\begin{align*}
\kappa_1(z) &= f(z) + z j'(z), \\
\kappa_2(z) &= h(z) - j(z)
\end{align*}
\]

and solve the differential equations

\[
\begin{align*}
\left( \frac{\kappa_1}{z^4} \right)' &= -\frac{\cos^3(\theta(z)) \sqrt{z^2 \theta'(z)^2} + 1}{3z^5} = -\frac{1 - m^2 z^2}{3z^5}, \\
\left( \frac{\kappa'_2}{z^3} \right)' &= 0.
\end{align*}
\]

The initial conditions are the same as for Minkowski embeddings, because the branes have to satisfy junction conditions at \( z_0 = 1/m \). Solving the differential equations yields

\[
\kappa_2(z) = 0
\]

and

\[
\kappa_1(z) = -\frac{z^4}{3} \int_{1/m}^z dz_1 \left( \frac{1}{z_1^5} - \frac{m^2}{z_1^3} \right) = -\frac{z^4}{12} \left[ -\frac{1}{z_1^4} + 2 \frac{m^2}{z_1^2} \right]_{1/m}
\]

\[
= \frac{1}{12} \left( 1 - m^2 z^2 \right)^2,
\]

which agrees with the known result. Therefore, the choice \( h(z) = j(z) \), which has been chosen due to symmetry considerations in the original calculation, is not just a gauge freedom, but is always satisfied. The solution for \( \kappa_1 \) is shown in Figure 4.1.
4. Flavour contribution to the entanglement entropy

4.2.2 Results for finite temperature

This section presents my calculation for the backreaction at finite temperature. Although the final calculation is mostly done numerically, let me first derive integral equations for the gauge-invariant combinations. Integrating the differential equations (4.20) under consideration of the appropriate initial conditions in equation (4.28) or (4.26) yields for $\kappa_1$

$$\kappa_1(z) = -z^4 \int_{z_0}^{\min(z,z_0)} \frac{\cos^3(\theta(z_1)) \sqrt{z_1^2 b(z_1) \theta'(z_1)^2 + 1}}{3z_1^3} \frac{dz_1}{\sqrt{z_1^2 b(z_1) \theta'(z_1)^2 + 1}}$$

(4.34)

and for $\kappa_2$

$$\kappa_2(z) = \frac{1}{12} b(z) \left\{ \begin{array}{ll} \kappa_1'(z_h) & \text{if } z_0 = z_h \\ 0 & \text{if } z_0 < z_h \end{array} \right.$$

(4.35)

The upper limits contain $\min(z,z_0)$ because the D7 branes only extend over the range $0 \leq z \leq z_0$. Therefore, the right-hand side of equations (4.20) implicitly contains Heaviside step functions.

The massless embedding is the trivial one with $\theta = 0$ and the backreaction can be solved analytically. For the gauge-invariant combinations, the results are

$$\kappa_1(z) = -z^4 \int_{z_h}^{z} \frac{dy}{3y^3},$$

$$= \frac{1}{12} b(z)$$

(4.36)

and

$$\kappa_2(z) = \frac{1}{12} b(z).$$

(4.37)
4.2 Backreaction of the metric

The result is shown in Figure 4.2.

For massive flavour, the analytical solution for the embedding is not known. Therefore, I solved the differential equations numerically. A summary of the numerical calculations can be found in section D.1 of the appendix. Figure 4.3 shows the results for different masses.

Let us analyse these plots qualitatively. For large mass, which corresponds to the small temperature limit, solutions are comparable to the zero temperature result. The gauge-invariant combinations are positive, non-increasing and have a vanishing derivative at the end of the probe branes. These are characteristic properties for all Minkowski embeddings, although for smaller mass the probe branes reach farer into the bulk and the backreaction differs more and more from the zero temperature result. Especially, \( \kappa_2 \) is positive and the boundary value depends on \( m/T \). Although it looks as if \( \kappa_2(0) \) is increasing with \( m/T \), we will later see that this is not true near the critical embedding, i.e. the embedding with \( \theta(z_h) = \pi/2 \). For black hole embeddings, the gauge-invariant combinations have a finite derivative at \( z_h \). It is interesting that for zero mass both gauge-invariant combinations agree.
Figure 4.3: Backreaction of metric

(a) $\kappa_1$ for Minkowski embeddings

(b) $\kappa_1$ for black hole embeddings

(c) $\kappa_2$ for Minkowski embeddings

(d) $\kappa_2$ for black hole embeddings
4.2 Backreaction of the metric

4.2.3 Metric in the field theory

In my calculation, I fixed the initial conditions in the bulk. However, the metric in the field theory is determined by the boundary behaviour of the bulk metric. To obtain it, we need to bring the metric in the Fefferman-Graham form

\[
g_{\text{AdS}} = \frac{L^2}{z^2} \left( (1 + t_0 f(z)) \frac{dz^2}{b} - b(1 + t_0 h(z)) dt^2 + (1 + t_0 j(z)) d\vec{x}^2 \right),
\]

(4.38a)

\[
= \frac{L^2}{y^2} \left( dy^2 + \hat{g}_{\mu\nu}(z, x) dx^\mu dx^\nu \right),
\]

(4.38b)

where \( \hat{g}(0, x) \) is the metric in the field theory. The new AdS radius is

\[
\tilde{L} = L \left( 1 + \frac{1}{2} t_0 f(0) \right) = L \left( 1 + \frac{1}{2} t_0 \kappa_1(0) \right),
\]

(4.39)

The new radial coordinate agrees at leading order with the old one

\[
y = z + \mathcal{O}(z^2),
\]

(4.40)

which simplifies the calculation of the field theory metric.

Therefore, the perturbed metric in the field theory is

\[
\hat{g}(0) = \begin{pmatrix}
-1 - t_0 h(0) + t_0 f(0) & 0 \\
0 & 1 + t_0 j(0) - t_0 f(0)
\end{pmatrix},
\]

(4.41)

where the boundary values of the metric perturbation are used. \( f(0) \) is fixed by equation (4.21). For a gauge which does not change the cut-off (c.f. calculation in section C of the appendix), we have \( j(0) = 1/12 \). Since the field theory metric should coincide with the Minkowski metric, the time component has to be rescaled

\[
t = \left( 1 - \frac{t_0}{2} \kappa_2(0) \right) \tau, \ t \in [0, \pi z_h],
\]

(4.42a)

\[
\tau \in \left[ 0, \pi z_h \left( 1 + \frac{t_0}{2} \kappa_2(0) \right) \right].
\]

(4.42b)

In the Euclidean formalism, the new time coordinate has a new period and a new associated temperature. When we want to obtain the backreacted metric with temperature \( T = (\pi z_h)^{-1} \), we have to add the flavour branes to a background with temperature \( \tilde{T} \)

\[
\tilde{T} = \frac{1}{\pi \tilde{z}_h},
\]

(4.43a)

\[
\tilde{z}_h = z_h + \delta z_h = z_h \left( 1 - \frac{t_0}{2} \kappa_2(0) \right).
\]

(4.43b)
Considering this, our previous initial values seem a bit arbitrary and in fact they are. For the Minkowski embeddings, we used junction conditions at the end of the branes. However, the metric perturbation does not have to end there, just the right hand sides of the differential equations (4.20) vanish for larger \( z \). The two gauge-invariant combinations are not independent but have to satisfy \( \kappa_1(z_h) = -\kappa_2(z_h) \), which means we need two fix two initial values. The straightforward way from the field theory side would be to fix the field theory metric (i.e. \( \kappa_2(0) = 0 \)) and the temperature, which is an initial condition at the horizon. There are however two major difficulties to this approach. The initial values are fixed at two different points, which is difficult to solve numerically. Additionally, the volume term would be contained in \( \delta_1 S_{EE} \) and had to be solved numerically, which would lead to inaccuracies. For that reason, I fixed the initial values in a way which makes the numerical result finite. I return to the physical setup by rescaling the time coordinate and shifting the temperature. The volume term is contained in the additional contribution \( \delta_2 S_{EE} \), which can be calculated analytically up to a constant.

Let us analyse the additional contribution due to the temperature shift. The previously mentioned Chang’s and Karch’s method determines the correction to the entanglement entropy for a fixed horizon position, i.e.

\[
\delta_1 S_{EE} = S_{EE}|_{z_h} - S_{EE}^{(0)}|_{z_h}
\]  

(4.44)

for a fixed width \( l \). However, we want the change for a fixed temperature, i.e.

\[
\delta S_{EE} = S_{EE}|_T - S_{EE}^{(0)}|_T
\]

\[
= S_{EE}|_{z_h+\delta z_h} - S_{EE}^{(0)}|_{z_h+\delta z_h} + S_{EE}^{(0)}|_{z_h+\delta z_h} - S_{EE}^{(0)}|_{z_h}.
\]

(4.45)

The first two terms reduce to \( \delta_1 S_{EE} \). The temperature change \( z_h \to z_h + \delta z_h \) only gives a higher order correction and can be neglected. The last two terms are

\[
\delta_2 S_{EE} = S_{EE}^{(0)}|_{z_h+\delta z_h}(z_*(l)) - S_{EE}^{(0)}|_{z_h}(z_*(l)),
\]

\[
= \frac{S_{EE}^{(0)}|_{z_h}(l)}{\delta z_h} \delta z_h,
\]

(4.46)

where \( z_*(l) \) is the inverse function of \( l(z_*) \). In chapter 3, I calculated the entanglement entropy as a function of the turning point \( z_\star \), but here the difference is calculated for a fixed width \( l \) and not for a fixed turning point.

Up to the numerical constant \( \kappa_2(0) \) in \( \delta z_h \), the additional contribution can be calculated analytically. The entanglement entropy for the horizon at \( z_h + \delta z_h \) can be written as

\[
S_{EE}^{(0)}|_{z_h+\delta z_h}(z_*(l)) = \frac{1}{(z_h + \delta z_h)^2} S_{EE}^{(0)}|_{z_h=1} \left( \frac{z_*(l)}{z_h + \delta z_h}, \frac{\epsilon}{z_h + \delta z_h} \right),
\]

(4.47a)

\[
\frac{z_*(l)}{z_h + \delta z_h} = z_\star|_{z_h=1} \left( \frac{l}{z_h + \delta z_h} \right).
\]

(4.47b)
The additional correction to the entanglement entropy is therefore
\[ \delta_2 S_{EE} = -\frac{\delta z_h}{z_h} \left( 2 S_{EE}^{(0)}|_{z_h}(z_*) + \frac{\partial S_{EE}^{(0)}|_{z_h}(z_*)}{\partial l(z_*)} l(z_*) + \partial_\epsilon S_{EE}^{(0)}|_{z_h}(z_*) \epsilon \right), \]
\[ = t_0 \kappa_2(0) \left( S_{EE}^{(0)}|_{z_h}(z_*) + \frac{1}{2} \partial_\epsilon S_{EE}^{(0)}|_{z_h}(z_*) \epsilon + \frac{1}{2} \partial S_{EE}^{(0)}|_{z_h}(z_*) \epsilon \right), \]
where \( \partial_\epsilon \) is short for the derivative after the turning point \( z_* \). The first two terms are the cut-off independent part of the entanglement entropy. Since the cut-off dependent part is independent of the width, this contribution is independent of the cut-off. The result is shown in Figure 4.4.

In the large-volume limit, \( S_{EE}^{(0)} \) is proportional to the width and the leading contribution to the correction \( \delta_2 S_{EE} \) is
\[ \delta_2 S_{EE} \propto \frac{3 t_0 \kappa_2(0)}{2} S \frac{\text{Vol}(B)}{\text{Vol}(\mathbb{R}^3)} + \cdots, \]
where \( S \) is the entropy of the adjoint degrees of freedom. Therefore, we obtain the expected volume term. This volume term is subtracted from \( \delta_2 S_{EE} \) to obtain the change of the ‘corrected’ entanglement entropy
\[ \delta_2 \tilde{S}_{EE} = t_0 \kappa_2(0) \left( \tilde{S}_{EE}^{(0)}|_{z_h}(z_*) + \frac{1}{2} \partial_\epsilon \tilde{S}_{EE}^{(0)}|_{z_h}(z_*) \epsilon + \frac{1}{2} \partial \tilde{S}_{EE}^{(0)}|_{z_h}(z_*) \epsilon \right). \]

In the following, we will first analyse the flavour correction to the thermal entropy, before returning to the change of the ‘corrected’ entanglement entropy.
4.3 Thermal entropy

The entanglement entropy does not only contain contributions due to entanglement but also contributions from the thermal entropy. Therefore, it is necessary to look at this contribution separately. Since our initial values are chosen such that the black hole entropy does not change through the backreaction, the characteristic volume term for the thermal entropy is contained in the contribution due to the rescaling of the temperature.

The entropy density of the adjoint degrees of freedom is

\[ s^{(0)} = \frac{N_c^2 \pi^2}{2} \text{Vol}(B) T^3. \]  

(4.51)

Using the result from equation (4.49), the flavour correction to the entropy density \( s \) is

\[ s = \frac{3}{4} N_f N_c \lambda \kappa_2(0) T^3. \]  

(4.52)

For a later comparison, we adapt the conventions from [38] and define

\[ \bar{N} = \frac{\lambda N_f N_c}{32} T^3. \]  

(4.53)

It is important to keep in mind that \( \kappa_2(0) \) as well as \( \bar{N} \) depend on the temperature. While the entropy density is proportional to \( \kappa_2(0) \), the behaviour of free energy density \( F \) and average energy \( \langle E \rangle \) is more complicated. Using the entropy density, it is possible to calculate the free energy density \( F \) and the on-shell action density of the D7 brane \( \mathcal{I}_{D7} = F/T \) by

\[
F(T) = - \int_0^T d\hat{T} \, s(\hat{T})|_{\bar{M}=\text{const.}},
\]

(4.54a)

\[
\mathcal{I}_{D7} = - \frac{F}{T}.
\]

(4.54b)

The average energy density is

\[
E = F + Ts,
\]

\[
= T \left( \mathcal{I}_{D7} + s \right). \tag{4.55}
\]

I used my results for the backreaction to interpolate \( \kappa_2(0) \) for general masses in units of temperature (i.e. \( \bar{M}/T \)). This gives the entropy density \( s(T) \). Numerically integrating this expression then yields \( F \) and \( E \). Figure 4.6 shows my results. They are equivalent to the results obtained by calculating the on-shell action, which are shown in Figure 4.7.

\[ ^3 \text{In the following, } s \text{ means only the flavour contribution. If we include also the contribution from the adjoint degrees of freedom, all thermodynamical quantities get shifted, but this shift is irrelevant for the discussion of the phase transition.} \]

\[ ^4 \text{We use } \bar{N} \text{ instead of } \mathcal{N} \text{ to avoid confusion.} \]
The stress energy tensor of the theory is

\[ \langle T_{\mu\nu} \rangle = \begin{pmatrix} E & -F \\ -F & -F \end{pmatrix}, \]

where \( P \) is the pressure. The trace of the stress energy tensor

\[ \text{Tr} \langle T_{\mu\nu} \rangle = -E - 3F = -T (s_{th} + 4 I_{DT}) \] (4.56)

is no longer vanishing, as can be seen in Figure 4.5. This means that the theory is no longer scale invariant.

Let us first look at this results from the gravity side. The interesting part is, that for the AdS-Schwarzschild metric, there are two topologically different classes of embeddings: the black hole and the Minkowski embedding. The first one contains the black hole, the second does not. The critical embedding is the limiting case, where the brane ends at the horizon with \( \theta(z_h) = \pi/2 \). It was already early suggested, that there is a phase transition between the different classes of embeddings [29]. Embeddings near the critical one show an oscillating behaviour and the quark condensate is no longer a single-valued function of the mass (compare Figure 2.5) [31,58]. The solution which minimizes the free energy is the physical one. My results in Figure 4.6 show that the phase transition occurs at a critical temperature, which is approximately \((T/\bar{M})_c = 0.7657\). They agree with results obtained by determining the on-shell action in [38], whose result are shown in Figure 4.7. In that paper, the phase transition occurs at \((T/\bar{M})_c = 0.765723\). It is a first order phase transition, because the first derivative of the free energy is discontinuous. When increasing the temperature, the embedding changes discontinuously from a Minkowski to a black hole embedding. Also the average energy and entropy increase discontinuously.

Let us turn to the field theory interpretation of this phase transition. It was shown in [30] that the behaviour of mesons (i.e. quark-antiquark bound states) is fundamentally
Figure 4.6: Flavour corrections to thermal entropy, on-shell action and energy density. The grey line marks the temperature at which the meson melting phase transition occurs: $T/\bar{M} = 0.765723$ [38]. The blue (dotted) line is the result for Minkowski embeddings and the red (dashed) line is for black hole embeddings.
4.3 Thermal entropy

Figure 4.7: Results from on-shell action
Plots taken from [38], labels modified
The red (solid) line is for black hole embeddings and the blue (dashed) line is for Minkowski embedding
different above and below the critical temperature. At low temperature, there is a
discrete meson spectrum with a mass gap and the mesons are stable, which is similar
to mesons of massive flavour at zero temperature. The flavour behave like a suspension
in the plasma, i.e. they are individual particles. In contrast, for high temperatures the
spectrum is continuous and gap-less. Mesons are described by quasi-normal modes
and are no longer stable. This means for temperatures above the critical temperature,
mesons melt and the energy dissipates into the plasma. Therefore, the flavour behave
like a solution in the plasma. There is an increase of the entropy at the phase transition.
It implies that there are degrees of freedom, which spontaneously get activated at the
critical temperature. However, this increase is small compared to the entropy difference
between the high and low temperature limit.

This phase transition makes it particularly interesting to look at the entanglement
entropy. In the previous chapter, we saw how the entanglement entropy of the adjoint
degrees of freedom is changed due to Debye screening. Therefore, we may suspect that
there is a qualitative difference in the flavour correction to this effect for Minkowski
and black hole embeddings.

4.4 ‘Corrected’ entanglement entropy

In the previous section, the flavour correction to the thermal entropy was calculated. In
this section, I calculate the flavour contribution to the ‘corrected’ entanglement entropy,
i.e. the entanglement entropy without the volume term from the thermal entropy.

The first contribution has to be calculated by using Chang’s and Karch’s method.
The first step is to calculate the ‘stress energy tensor’ of the minimal surface, as defined
in equation (4.4). For a straight belt, the minimal area in the unperturbed geometry
is given in equation (3.12) and the square root of the induced metric is

\[
\sqrt{\gamma} = \sqrt{g_{11}(z)g_{22}(z)g_{33}(z) \left( \frac{dz}{dx^1} \right)^2 \frac{g_{zz}(z)}{g_{11}(z)} + 1},
\]  

(4.57)

The differential equation for \( z \) is given in equation (3.14)

\[
\frac{\partial z}{\partial x^1} = \sqrt{b(z) \frac{z^3}{z^*}} \sqrt{1 - \frac{z^6}{z^*}}.
\]  

(4.58)

Calculating the ‘stress energy tensor’ \( T^{MN}_{\text{min}} \) for the integral terms in \( A \) (c.f. equation (3.12)) yields

\[
T^{ab}_{\text{min}} = \frac{2}{\sqrt{\gamma}} \frac{\partial \sqrt{\gamma}}{\partial g_{ab}} = \frac{z^2}{L^2} \begin{pmatrix}
(1 - \frac{z^6}{z^*}) & b(z) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{z^6}{z^*} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}^{ab},
\]  

(4.59)
where the coordinates are in the order \((z, t, x^i)\). Using the ansatz for the metric perturbation

\[
\delta g_{\text{AdS}} = \frac{L^2}{z^2} \left( f(z) \frac{dz^2}{b} - h(z) b dt^2 + j(z) d\vec{x}^2 \right)
\]

the first order flavour correction to the entanglement entropy is

\[
\delta S_{EE} = \frac{t_0 L^3 \tilde{t}^2}{4G_N} \int_0^{\min(z_0, z_\star)} \frac{dz}{z^3 \sqrt{b(z)}} \left( \sqrt{1 - \frac{z^6}{z_\star^6}} f(z) + \frac{j(z)}{\sqrt{1 - \frac{z^6}{z_\star^6}}} \left( 2 + \frac{z^6}{z_\star^6} \right) \right). \tag{4.61}
\]

We can replace \(f(z)\) with

\[
f(z) = \frac{1}{b(z)} \kappa_1(z) - z \sqrt{b} \left( \frac{j(z)}{b(z)} \right)', \tag{4.62}
\]

where I used equation (4.16a) and rewrote the terms containing \(j\). This allows to partially integrate and the integrand becomes gauge-independent. The result is

\[
\delta S_{EE} = \frac{t_0 L^3 \tilde{t}^2}{4G_N} \left( \frac{1}{2} j''(0) + \frac{j'(0)}{\epsilon} + \frac{j(0)}{\epsilon^2} \right) + \frac{t_0 L^3 \tilde{t}^2}{4G_N} \int_0^{\min(z_0, z_\star)} \frac{dz}{z^3 b(z)^{3/2}} \frac{\sqrt{1 - \frac{z^6}{z_\star^6}}}{\kappa_1(z)} \kappa_1(z). \tag{4.63}
\]

For gauge-dependent part, we fix a gauge in which the flavour branes leave the cut-off unchanged. The corresponding boundary expansion of \(j\) is given in equation (C.9). The gauge-dependent part is therefore

\[
\delta S_{EE,G.D.} = \frac{t_0 L^3 \tilde{t}^2}{4G_N} \left[ - \frac{m^2}{18} + \frac{1}{12 \epsilon^2} \right]. \tag{4.64}
\]

The gauge-independent part is

\[
\delta S_{EE,G.I.} = \frac{t_0 L^3 \tilde{t}^2}{4G_N} \int_0^{\min(z_0, z_\star)} \frac{dz}{z^3 b(z)^{3/2}} \kappa_1(z). \tag{4.65}
\]

Combining this with the results for the contribution due to the temperature shift in equation (4.50), the total flavour correction to the ‘corrected’ entanglement entropy is

\[
\delta \tilde{S}_{EE} = \delta_1 S_{EE} + \delta_2 \tilde{S}_{EE}. \tag{4.66}
\]

### 4.4.1 Review of zero temperature results

Let us check my previous calculation by applying it to the zero temperature case. This was already solved in [35].
4. Flavour contribution to the entanglement entropy

The calculations in the previous sections already showed that the turning point is $z_0 = 1/m$, the width of the belt is

$$l(\tilde{z}) = \frac{2\sqrt{\pi}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})} \tilde{z}$$

(4.67)

and the backreaction of the metric is

$$\kappa_1 = \frac{1}{12} \left(1 - m^2 \tilde{z}^2\right)^2,$$

(4.68a)

$$\kappa_2 = 0.$$  

(4.68b)

This means in particular that the only non-vanishing contribution is $\delta_1 S_{EE}$ and the gauge-independent part is

$$\delta S_{EE,G.I.} |_{T=0} = t_0 \frac{L^3 \tilde{l}^2}{4G_N} \int_{0}^{\min(z_0, \tilde{z})} dz \frac{\sqrt{1 - \frac{z^6}{\tilde{z}^6}}}{z^3} \kappa_1(z),$$

$$= t_0 \frac{L^3 \tilde{l}^2}{48G_N} \int_{0}^{\min(z_0, \tilde{z})} dz \frac{\sqrt{1 - \frac{z^6}{\tilde{z}^6}}}{z^3} \left(1 - \frac{z^2}{z_0^2}\right)^2,$$

$$= t_0 \frac{L^3 \tilde{l}^2}{48G_N z_0^2} \int_{0}^{\min(z_0/\tilde{z}, 1)} ds \frac{\sqrt{1 - s^6}}{s^3} \left(1 - 2s^2 \frac{\tilde{z}^2}{z_0^2} + s^4 \frac{z_0^4}{\tilde{z}^4}\right).$$

(4.69)

The integrals can be solved using $I_2$ from section B.2.2.

The result is

$$\delta S_{EE,G.I.} |_{T=0} = - \frac{t_0 L^3 \tilde{l}^2}{288G_N} \left[ -\frac{3}{\epsilon^2} + 12m^2 \log \left( \frac{\min(1, m\tilde{z})}{m\epsilon} \right) \right]$$

$$- 3m^2 \min(1, m\tilde{z})^2 \ _2F_1 \left( \left\{ \frac{1}{2}, \frac{1}{3} \right\}; \left\{ \frac{4}{3} \right\}; \min(1, \frac{1}{m\tilde{z}})^6 \right)$$

$$+ \frac{3m^2}{\min(1, m\tilde{z})^2} \ _2F_1 \left( \left\{ \frac{1}{2}, \frac{1}{3} \right\}; \left\{ \frac{2}{3} \right\}; \min(1, \frac{1}{m\tilde{z}})^6 \right)$$

$$- \frac{1}{m^4 \tilde{z}^6} \min(1, m\tilde{z})^6 \ _3F_2 \left( \left\{ \frac{1}{2}, 1, 1 \right\}; \left\{ 2, 2 \right\}; \min(1, \frac{1}{m\tilde{z}})^6 \right).$$

(4.70)

The gauge-dependent term is temperature independent and hence

$$\delta S_{EE,G.D.} |_{T=0} = \frac{t_0 L^3 \tilde{l}^2}{4G_N} \left[ -\frac{m^2}{18} + \frac{1}{12\epsilon^2} \right].$$

(4.71)

The function is piece-wise defined, depending on the value of the $m\tilde{z}$. For small mass (i.e. $m\tilde{z} \leq 1$), the result is

$$\delta S_{EE} |_{T=0} = \frac{t_0 L^3 \tilde{l}^2}{48G_N} \left( \frac{3}{2\epsilon^2} + \frac{\sqrt{\pi} \Gamma(-1/3)}{12\tilde{z}^2\Gamma(7/6)} + \frac{\sqrt{\pi} m^4 \tilde{z}^2 \Gamma(1/3)}{12\Gamma(11/6)} + 2m^2 \ln \left( \frac{\epsilon}{\tilde{z}} \right) - \frac{2m^2}{3} \ln 2 \right).$$

(4.72)
The massless result can be obtained by taking the limit $m \to 0$ and is
\[
\delta S_{EE|T=0} = \frac{t_0 L^3 \ell^2}{48 G_N} \left( \frac{3}{2 \epsilon^2} + \frac{\sqrt{\pi} \Gamma (-1/3)}{12 \epsilon^2 \Gamma (7/6)} \right) = \frac{t_0 S_{EE}^{(0)}|T=0}{8 }.
\] (4.73)

For large mass (i.e. $m \tilde{z} > 1$), the result is
\[
\delta S_{EE|T=0} = - \frac{t_0 L^3 \ell^2}{288 G_N} \left[ - \frac{9}{\epsilon^2} - 12 m^2 \log (m \epsilon) + 4 m^2 - 3 m^2 \right] _2 F_1 \left( \left\{ - \frac{1}{2}, \frac{1}{3} \right\}; \left\{ \frac{4}{3} \right\}; \frac{\tilde{z}^6}{2^6} \right)
+ 3 m^2 F_1 \left( \left\{ - \frac{1}{2}, - \frac{1}{3} \right\}; \left\{ \frac{2}{3} \right\}; \frac{z^6}{2^6} \right) - \frac{1}{m^4 \tilde{z}^6} 3 F_2 \left( \left\{ \frac{1}{2}, 1, 1 \right\}; \left\{ 2, 2 \right\}; \frac{z^6}{2^6} \right).
\] (4.74)

For heavy flavours, the leading order contributions in $1/m$ are
\[
\delta S_{EE|T=0} = \frac{t_0 L^3 \ell^2}{48 G_N} \left( \frac{3}{2 \epsilon^2} + 2 m^2 \log(m \epsilon) - \frac{2}{3} m^2 - \frac{1}{48 m^4 \tilde{z}^6} + \mathcal{O} \left( \frac{1}{m^{10} \tilde{z}^{12}} \right) \right).
\] (4.75)

An important feature of this result the asymptotic value for $l \to \infty$
\[
\delta S_{EE|T=0} = \frac{t_0 L^3 \ell^2}{48 G_N} \left( \frac{3}{2 \epsilon^2} + 2 m^2 \log(m \epsilon) - \frac{2}{3} m^2 \right).
\] (4.76)

Intuitively, it is convenient to write the flavour correction to the entanglement entropy as
\[
\delta S_{EE|T=0} = m^2 \delta S_{EE|T=0} (1, \epsilon m, \tilde{z}, m).
\] (4.77)

However, the problem is in the ambiguity of the cut-off dependent term. When considering
\[
\frac{\delta S_{EE|T=0}}{m^2} = - \frac{t_0 L^3 \ell^2}{96 G_N m^2} \left[ \frac{3}{\epsilon^2} + 4 m^2 \log (\epsilon m) \right],
\] (4.78)
the cut-off independent part is a function only of $\tilde{z} m$. The result for this cut-off independent part of $\delta S_{EE}$ and the entropic c-function are shown in Figure 4.8.

However, this choice does not work well for the finite temperature result. To be able to compare this results to finite temperature results, we introduce $z_h$, which is determined by the finite temperature theory. The flavour correction to the entanglement entropy can be written as
\[
\delta S_{EE|T=0} = \frac{1}{z_h^2} \delta S_{EE|T=0} (m z_h, \frac{\epsilon}{z_h}, \frac{\tilde{z}}{z_h}).
\] (4.79)

To keep this scaling behaviour, we will in the following work with the cut-off dependent part
\[
\delta S_{EE|T=0} = - \frac{t_0 L^3 \ell^2}{96 G_N} \left[ \frac{3}{\epsilon^2} + 4 m^2 \log \left( \frac{\epsilon}{z_h} \right) \right].
\] (4.80)
4. Flavour contribution to the entanglement entropy

Up to the prefactor $1/z_h^2$, it only depends on $m/T$ and $\tilde{z}/z_h$. Especially, the asymptotic constant is

$$
\delta S_{EE}|_{T=0}(\infty) - \frac{t_0 L^3}{96 G_N} \left[ \frac{3}{\epsilon^2} + 4 m^2 \log \left( \frac{\epsilon}{z_h} \right) \right] = \frac{t_0 L^3}{48 G_N} \left( 2 m^2 \log (m z_h) - \frac{2}{3} m^2 \right). \tag{4.81}
$$

The result of this cut-off independent part is shown in Figure 4.11.

4.4.2 Results for finite temperature

In the previous section, we saw that my result reproduces the known flavour correction to the entanglement entropy at zero temperature. In this section, I generalize this calculation to finite temperature. In contrast to the calculation at zero temperature, the flavour correction to the entanglement entropy has to be determined numerically.

For the numerical calculation, the divergent cut-off dependent terms of $\delta S_{EE}$ have to be separated. The gauge-independent part $\delta_1 S_{EE,G.I.}$ splits into a divergent part and a finite part. The divergent part can be removed from the integrand and integrated separately. This yields

$$
\delta_1 S_{EE,G.I.} = \frac{t_0 L^3}{4 G_N} \left[ \frac{1}{24 \epsilon^2} + m^2 \log(\epsilon) - \frac{m^2}{6} \log(\min(z_0, z_*) \right) - \frac{1}{24 \min(z_0, z_*)^2} 
+ \int_0^{\min(z_0, z_*)} \frac{dz}{z^3 b(z)^{3/2}} \kappa_1(z) - \frac{\kappa_1(0)}{z^3} - \frac{\kappa_1'(0)}{z^2} - \frac{\kappa_1''(0)}{2z} \right]. \tag{4.82}
$$

The flavour correction to the entanglement entropy due to Karch’s and Chang’s method is

$$
\delta_1 S_{EE}(B) = \delta_1 S_{EE,G.I.} + \delta_1 S_{EE,G.D.}, \tag{4.83}
$$
which is after using the equations (4.82) and (4.64)

\[
\delta_1 S_{EE} = \frac{t_0L^3 \ell_0^2}{96G_N} \left[ \frac{3}{\ell^2} + 4m^2 \log \left( \frac{\epsilon}{z_h} \right) - \frac{4}{3} m^2 - \frac{1}{\min(z_0, z_*)^2} \right] + 4m^2 \log \left( \frac{z_h}{\min(z_0, z_*)} \right) \\
+ \frac{t_0L^3 \ell_0^2}{4G_N} \int_0^{\min(z_0, z_*)} dz \left( \frac{1}{z^3 b(z)^{3/2}} \kappa_1(z) - \frac{\kappa_1(0)}{z^3} - \frac{\kappa_1'(0)}{z^2} \right).
\]

(4.84)

It depends on \(\epsilon/z_h, \ z_0/z_h\) and the temperature due to an overall prefactor.

\[
\delta_1 S_{EE}(B) = \frac{t_0L^3 \ell_0^2}{96G_N z_h^2} \left[ \frac{3z_h^2}{\ell^2} + 4z_h^2 m^2 \log \left( \frac{\epsilon}{z_h} \right) - \frac{4z_h^2}{3} m^2 - \frac{z_h^2}{\min(z_0, z_*)^2} \right] \\
- 4z_h^2 m^2 \log \left( \frac{\min(z_0, z_*)}{z_h} \right) + \frac{t_0L^3 \ell_0^2}{4G_N z_h^2} \int_0^{\min(z_0, z_*)} z_h dz \left( \\
\frac{\sqrt{1 - \frac{s^6}{z_h^6}}}{z_h^{3/2}} \kappa_1|_{z_h=1}(z) - \frac{\kappa_1|_{z_h=1}(0)}{z^3} - \frac{\kappa_1'|_{z_h=1}(0)}{z^2} - \frac{\kappa_1''|_{z_h=1}(0)}{2z} \right).
\]

(4.85)

Deducing the same cut-off dependent term as before, the cut-off independent term

\[
\delta S_{EE} = \frac{t_0L^3 \ell_0^2}{96G_N} \left[ \frac{3}{\ell^2} + 4m^2 \log \left( \frac{\epsilon}{z_h} \right) \right]
\]

(4.86)

and depends up to the prefactor \(1/z_h^2\) only on \(z_*/z_h\) and \(z_0/z_h\).

Both the zero and the finite temperature result are divergent in the small-volume limit. The divergent terms can be determined analytically. The divergent terms of the area change are

\[
\delta A_{G.I.} = t_0L^3 \ell_0^2 \int_0^{z_*} dz \frac{\sqrt{1 - \frac{s^6}{z_*^6}}}{z_*^{3/2}} \kappa_1(z),
\]

\[
= t_0 \frac{L^3 \ell_0^2}{\kappa_*^2} \int_0^{1} ds \int_0^{z_*} \frac{\sqrt{1 - \frac{s^6}{z_*^6}}}{s^3 b(s z_*)^{3/2}} \kappa_1(s z_*),
\]

\[
= t_0L^3 \ell_0^2 \int_0^{1} ds \frac{\sqrt{1 - \frac{s^6}{z_*^6}}}{12z_*^3} \left( 2 - 2s^2 z_*^3 \theta(0)^2 \right) + O(z_*^2).
\]

(4.87)

This integral can be solved analytically. The stammfunction for an integral of type \(\sqrt{1 - \frac{s^6}{z_*^6}}\) is given in subsection B.2.2 of the appendix. Hence, the result for \(\delta S_{G.I.}\) is

\[
\delta_1 S_{G.I.} = \frac{t_0L^3 \ell_0^2}{96G_N \epsilon^2} + \frac{t_0L^3 \ell_0^2 m^2}{24G_N} \ln \left( \frac{\epsilon}{z_*} \right) - \frac{t_0\sqrt{\pi} L^3 \Gamma \left( \frac{3}{2} \right) \ell_0^2}{192G_N z_*^2 \Gamma \left( \frac{3}{2} \right)}
+ \frac{t_0L^3 \ell_0^2 \theta'(0)^2}{72G_N} + O(z_*^2).
\]

(4.88)
4. Flavour contribution to the entanglement entropy

The red (dashed) line is for black hole embeddings and the blue (dotted) line is for Minkowski embeddings.

We saw in equation (3.17) that the temperature correction to the turning point is of order $O(z_*/z_h)^4$. Also, the divergent part of the entanglement entropy $S_{EE}^{(0)}$ in equation (3.23) shows that $\delta_2 S_{EE}$ is of order $O(l^2)$ and does not yield a contribution to the divergent terms. Therefore, the leading contributions in the limit $l \to 0$ are temperature independent and hence agree for zero and for finite temperature.

The numerical calculations are presented in section D.2 of the appendix. Figure 4.10 shows the results for different masses.

In the limit $l \to \infty$, the entanglement entropy approaches a constant. For the zero temperature result, the analytical expression for the constant is given in equation (4.81). For the finite temperature result, this constant has to be determined from the numerical solutions. The results are shown in Figure 4.9.
Figure 4.10: Flavour correction to the entanglement entropy for finite temperature in units of $\lambda N_c N_f T^2 \delta^2$

Figure 4.11: Flavour correction to the entanglement entropy for zero temperature in units of $\lambda N_c N_f T^2 \delta^2$
4.5 Discussion

In this chapter, I have calculated the leading order flavour correction to the entanglement entropy. The expansion parameter $t_0$ is proportional to

$$t_0 \propto \lambda \frac{N_f}{N_c} = g_{YM}^2 N_f.$$  \hfill (4.89)

Therefore, expansion in $t_0$ is an expansion in the number of flavour loops (c.f. Figure 4.12). This means that the probe approximation (i.e. $t_0 = 0$) is the ‘quenched’ approximation, where contributions from flavour loops are neglected. Therefore, I have calculating the leading order correction to the entanglement entropy of the adjoint degrees of freedom due to flavour loop corrections. This flavour correction is proportional to

$$\delta S_{EE} \propto \frac{t_0 L^3}{G_N} \propto \lambda N_f N_c = g_{YM}^2 N_f N_c^2.$$  \hfill (4.90)

Consequently, this result cannot be compared to results at weak coupling. The same is of course also valid for the thermal entropy. I have not calculated the flavour correction due to the new degrees of freedom of the mesons, which is proportional to $N_f^2 N^2$. This kind of contribution is of second order in $t_0$

$$N_f^2 = N_c^2 \frac{1}{\lambda^2} (\lambda N_f / N_c)^2 \propto \frac{1}{\lambda^2} N_c^2 t_0^2$$  \hfill (4.91)

and cannot be obtained by Chang’s and Karch’s method.

On the field theory side, the flavour degrees of freedom are added as two $N = 1$ chiral multiplets $\tilde{Q}^r, Q_r, r \in 1, \ldots, N_f$. They can be massive and couple to $\Phi_3$, which is one of the $N = 1$ chiral multiplets contained in the $N = 4$ gauge multiplet. The new Lagrangian is

$$\mathcal{L} = \mathcal{L}_{SYM} + \text{kinetic term} + \int d^3 \theta \tilde{Q}^r(m_q + \Phi_3)Q^r + \text{c.c.},$$  \hfill (4.92)

where $\mathcal{L}_{YM}$ is the Lagrangian of $N = 4$ SYM theory. The coupling to the chiral multiplet $\Phi_3$ is a marginal deformation and the mass term is a relevant deformation of the conformal field theory.
In the following, I will first look at the cut-off dependent terms and then consider cut-off independent contribution to the entanglement entropy. Since $\delta_2 S_{EE}$ is cut-off independent, the relevant terms are due to $\delta_1 S_{EE}$. My result in equation (4.84) shows that these terms are temperature independent. The temperature is not a parameter describing the conformal field theory, but a parameter describing the excited state of the theory. In quantum field theory, universal terms of the entanglement entropy, like the logarithmic cut-off dependent term, are thought to be universal (i.e. state independent) although a proof of this is lacking. However, it was shown holographically that all cut-off divergent terms are universal [50]. Moreover, the origin of the cut-off dependent terms is clear. Adding flavour shifts the AdS radius to

$$L \rightarrow L \left(1 + \frac{t_0}{24}\right). \quad (4.93)$$

The $1/\epsilon^2$ term of the entanglement entropy $S_{EE}^{(0)}$ is proportional to $L^3$, hence the area term is rescaled

$$\frac{L^3 \tilde{R}^2}{4G_N \epsilon^2} \rightarrow \frac{L^3 \tilde{R}^2}{4G_N \epsilon^2} \left(1 + \frac{t_0}{8}\right) = \frac{N_c^2 \tilde{R}^2}{2\pi \epsilon^2} + \frac{\lambda N_f N_c \tilde{R}^2}{2\pi^3 \epsilon^2}. \quad (4.94)$$

For massive flavour, there is an additional logarithmic term. This term is characteristic for a relevant deformation of a conformal field theory. The holographic result agrees with the field theory result, see [35,59].

For zero mass, there is only a marginal deformation of the field theory. It was already observed for the zero temperature case that the correction to the entanglement entropy is proportional to the entropy without flavours, explicitly

$$\delta S_{EE} = \frac{t_0}{8} S_{EE}^{(0)}. \quad (4.95)$$

Our numerical results show that for finite temperature, the flavour correction to the entanglement entropy is proportional to the corresponding (i.e. finite temperature) entropy without flavours. This applies to the ‘corrected’ entanglement entropy as well as to the thermal entropy. The entanglement entropy without flavours is proportional to $N_c^2$. Adding massless flavour is therefore equivalent to considering the entanglement entropy for a theory with an increased number of colours

$$N_c \rightarrow N_c \left(1 + \frac{t_0}{16}\right),$$

$$= N_c + \lambda \frac{16\pi}{N_c} N_c N_f. \quad (4.96)$$

On the gravity side, this appears as a shift of the AdS radius, as was already noticed in equation (4.39).
For massive flavour, my results in Figure 4.10 look similar to the zero temperature results in Figure 4.11. The difference
\[
\Delta \delta S_{EE} = \delta S_{EE} - \delta S_{EE}|_{T=0}
\]
(4.97)
is plotted in Figure 4.13. For the Minkowski embeddings, they are so small that already effects due to numerical inaccuracy are visible. Of special interest is the comparison of the behaviour for the limit of a small and a wide belt. The divergent terms for a small width agree to the result at zero temperature (c.f. equation (4.88)). On the field theory side, this implies that both theories approach the same UV fixed point with the same effective degrees of freedom. For a wide belt, the ‘corrected’ entanglement entropy approaches a constant. This constant differs to the zero temperature result, as can be seen in Figure 4.9. This means that there is an additional area law. This implies that there is a change in the correlation length due to Debye screening. For temperatures far below the ‘meson melting’ phase transition, the finite temperature result agrees with the zero temperature result. Even for temperatures near the phase transition, there is only a slight positive deviation, which implies that the flavour correction reduces the effect of the Debye screening slightly. However, this changes drastically for temperatures above the phase transition. For zero temperature, the asymptotic constant has a minimum and then approaches zero in the massless limit. For finite temperature however, the asymptotic constant is further decreasing when approaching the massless limit and differs significantly from the zero temperature result.

In [55], it was found that the thermally induced masses are in the range $0.5T < \bar{M}_{adj} < 1.5T$. If the flavour mass is significantly larger than this, the induced masses can be neglected, which explains the agreement with the zero temperature result for large mass. The reduced Debye screening near the phase transition could mean that the effective mass of the flavour is less than the effective mass of the adjoint degrees of freedom and the flavour enhance long range correlations. An additional effect would be a thermally induced mass increase of the flavour, which would decrease long range correlations. My results indicate that this effect becomes dominant above the critical temperature, i.e. when the mesons melt, but not for stable mesons. However, further field theory results for the relationship between correlation length and area law are needed for a deeper physical understanding of these results. In particular, field theory results for the entanglement entropy of free, massive fields would give an interesting comparison.
Figure 4.13: Difference of entropy correction to zero temperature result
4. Flavour contribution to the entanglement entropy
Chapter 5

Conclusions and outlook

The purpose of this work was to examine entanglement in $\mathcal{N} = 4$ SYM theory. The entanglement entropy is a measure for the quantum entanglement between a region and its complement. It is an important order parameter for quantum phase transitions and a measure for the effective degrees of freedom. However, there are only a few field theory results for higher dimensions. Fortunately, the AdS/CFT correspondence allows us to obtain the entanglement entropy at strong coupling by calculating it holographically. Ryu’s and Takayanagi’s proposal simplifies the complicated quantum calculation on the field theory side to the calculation of the minimal surface area in AdS. My findings split in two parts: the calculation of the entanglement entropy of the adjoint degrees of freedom and the calculation of the leading order flavour correction to the entanglement entropy.

Let us first focus on the result at finite temperature without flavour. For zero temperature, the entanglement entropy of $\mathcal{N} = 4$ SYM theory was already determined in [19]. The conformal symmetry determines this result already up to two proportionality constants. At finite temperature, there is an additional length scale in the theory, which makes the result less predictable. For a two dimensional field theory, the holographic entanglement entropy at finite temperature was already solved analytically in [18, 19], but higher dimensional cases where only considered in the large-volume limit. In this limit, the dominant contribution of the entanglement entropy is from the thermal entropy. Quantitative calculations for higher-dimensional cases were only performed numerically up to now [37]. The new result of this thesis is the analytical calculation of the entanglement entropy for $\mathcal{N} = 4$ SYM theory (i.e. a four-dimensional field theory) at finite temperature. The solution agrees with the numerical result and reproduces the correct thermal entropy. It shows an additional area law, which indicates a mass gap in the theory. This phenomenon is well-known and called Debye screening. It was already observed in [23] that the adjoint degrees of freedom obtain a thermally induced mass. The associated finite correlation length causes a screening of test-charges in the plasma. Furthermore, I used my result for the entanglement entropy to calculate the entropic c-function, which shows the effect of the Debye screening on the effective degrees of freedom. In the UV, the finite temperature state approaches the same UV fixed point as the vacuum state and has the same degrees of freedom.
At finite energy however, the number of effective degrees of freedom is reduced. In the IR, the state approaches a theory where all degrees of freedom are heavy and can be integrated out, whereby the IR theory has no effective degrees of freedom. Unfortunately, there are no field theory results for the entanglement entropy of free, massive fields. Otherwise, it would be possible to compare this area law to the free field theory result obtained by using the thermally induced masses.

The advantage of the analytical result is that it can be calculated with arbitrary precision and can be easily generalized to the $d + 1$-dimensional background $\text{AdS}_{d+1}$. Generalizations of the AdS/CFT duality connect this background to a dual field theory CFT$_d$ in $d$ dimensions. Especially interesting are the dualities $\text{AdS}_3$/CFT$_2$, $\text{AdS}_4$/CFT$_3$ and $\text{AdS}_7$/CFT$_6$. The entanglement entropy for $\text{AdS}_3$/CFT$_2$ is already known, so it would be interesting to see how the general results simplifies. $\text{AdS}_4$/CFT$_3$ is dual to $\mathcal{N} = 8$ CFT, which is of interest in condensed matter physics. For $\text{AdS}_7$/CFT$_6$, the dual field theory is the $\mathcal{N} = (2, 0)$ CFT. [8, 11]

In the second part of my calculations, I have determined the leading order flavour contribution to the entanglement entropy. Flavour degrees of freedom are introduced on the gravity side in form of probe $D_p$-branes. In [33], Chang and Karch developed a simple method to calculate the leading order correction to the entanglement entropy. This method is restricted to probe branes which do not source the five-form field strength. Therefore, it can be applied to probe $D7$-branes with $F \wedge F = 0$, which are flavour degrees of freedom on the field theory side. This flavour are introduced as two chiral multiplets and couple to the adjoint degrees of freedom. It is important to keep in mind that Chang’s and Karch’s method calculates only the leading order correction in

$$\lambda \frac{N_f}{N_c} = g_Y^2 N_f, \quad (5.1)$$

i.e. these are corrections to the entanglement entropy of the adjoint degrees of freedom due to flavour loop corrections. The results are not comparable to weak coupling results. This method was already applied to massive flavour at zero temperature [35] and massless flavour at finite temperature and charge density in [36].

In contrast, I have considered flavour at zero charge density but at finite mass and temperature. As expected, the entanglement entropy approaches the thermal entropy in the large-volume limit. This allowed me to calculate the thermal entropy, the average energy and the free energy. It reproduced results from [38] and showed the ‘meson melting’ phase transition. A further important result was the influence of the flavour on the area term, which I found in the previous part of my results. There are several phenomenons which could influence the Debye screening. It is possible that the induced mass of the adjoint degrees of freedom is larger than the effective flavour mass, whereby flavour enhance long range correlations and weaken the effect of the Debye screening. My result show a slight increase for flavour around the critical temperature. However, also the effective flavour mass is thermally increased, which would increase the effect of the Debye screening. It is interesting that this appears to have a large effect for melted mesons at temperatures above the phase transition, but not for stable mesons.
This part of the results show that the entanglement entropy due to flavour is a useful tool to investigate phase transitions of the theory. Karch’s and Chang’s method can be used for probe branes which do not source the five-form field-strength. D7-branes are restricted to $F \wedge F = 0$, but $F$ does not have to vanish. In particular, it is possible to introduce a finite charge density by introducing a finite $A_t(z)$ and thereby generalizing the results in [36] to massive flavour. A different setup are probe D5-branes. Here, the restriction is $F = 0$. This new degrees of freedom do not correspond to flavour on the field theory side, because they only propagate in two spatial directions. This defect field theory is of interest in condensed matter physics. For these cases, only the backreaction of the metric is changed. The remaining part of the calculation agrees one-to-one with my calculation. Furthermore, it would be interesting to generalize Karch’s and Chang’s method to probe branes which source the five-form field strength, for example D7-branes with charge density and magnetic field. This system has, depending on the temperature, either one or two phase transitions, which can also be of second order [60].
5. Conclusions and outlook
Appendix A

Conventions and AdS/CFT dictionary

A.1 Conventions

We are working in units where
\[ c = k = \hbar = 1. \] (A.1)

A.1.1 Indices

During this thesis, we use double index notation without explicitly writing the sum. We use different types of letter for different coordinates.

<table>
<thead>
<tr>
<th>M, N</th>
<th>{z, 0, 1, 2, 3}</th>
<th>AdS coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>\mu, \nu</td>
<td>{0, 1, 2, 3}</td>
<td>spacetime coordinates</td>
</tr>
<tr>
<td>i, j</td>
<td>{1, 2, 3}</td>
<td>spatial coordinates</td>
</tr>
<tr>
<td>a, b</td>
<td>{0, \ldots, 9}</td>
<td>all coordinates (AdS$_5 \times S^5$)</td>
</tr>
</tbody>
</table>

A.1.2 Metric

The metric for AdS$_5 \times S^5$ is
\[ ds^2 = \frac{L^2}{z^2} \left( \frac{dz^2}{b(z)} - b(z) dt^2 + d\vec{x}^2 \right) + L^2 d\Omega_5^2, \] (A.2a)
\[ d\Omega_5^2 = d\theta^2 + \sin^2 \theta \ d\psi^2 + \cos^2 \theta \ d\Omega_3^2 \] (A.2b)
with
\[ b(z) = 1 - \frac{z^4}{z_h^4}. \] (A.3)

The ansatz for the metric perturbation is
\[
g_{\text{AdS}} = \frac{L^2}{z^2} \left( \frac{dz^2}{b(z)} - b(z) dt^2 + d\vec{x}^2 \right) + t_0 \delta g_{\text{AdS}}, \tag{A.4a}
\]
\[
\delta g_{\text{AdS}} = \frac{L^2}{z^2} \left( f(z) \frac{dz^2}{b} - h(z) b dt^2 + j(z) d\vec{x}^2 \right). \tag{A.4b}
\]

### A.2 AdS/CFT dictionary

- **5d Newton’s constant** $G_N$
  \[ G_N = \frac{\pi L^3}{2 N_c^2} \]
- **AdS-radius** $L$
  \[ L^4 = 2 \lambda \alpha'^2 \]
- **t’Hooft coupling** $\lambda$
  \[ \lambda = g_Y^2 N_c = 2 \pi g_s N_c \]
- **Horizon** $z_h$
  \[ z_h = \frac{1}{\pi T} \]
- **D7-brane tension** $T_7$
  \[ T_7 = \frac{\lambda N_c}{16 \pi^6 L^8} \]
- **Expansion parameter** $t_0$
  \[ t_0 = \frac{\lambda N_f}{\pi^2 N_c} \]
- **Mass** $\tilde{M}$
  \[ \frac{\tilde{M}}{T} = \frac{\sqrt{2}}{\pi} \theta'(0) \]
- **Quark condensate** $c$
  \[ c = \frac{\sqrt{2}}{3 \pi^3 T^3} \left( \theta'''(0) - \theta'(0)^3 \right) \]
Appendix B

Generalized hypergeometric functions

B.1 Definitions

A generalized hypergeometric function\(^1\) is defined as

\[
pF_q\left(\{a_1, \ldots, a_p\}; \{b_1, \ldots, b_q\}; z\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{m=1}^{p} (a_m)_{(n)} \frac{1}{\prod_{m=1}^{q} (b_m)_{(n)}} z^n, \quad (B.1a)
\]

\[
= \sum_{n=0}^{\infty} c_n z^n. \quad (B.1b)
\]

It can be reconstructed from the coefficients by calculating the ratio

\[
\frac{c_{n+1}}{c_n} = \frac{\prod_{m=1}^{p} (a_m + n)}{\prod_{m=1}^{q} (b_m + n)} \frac{1}{m+1}. \quad (B.2)
\]

Some special cases are

\[
\log(1 + z) =_{z} 2F_1(\{1, 1\}; \{2\}; -z), \quad (B.3a)
\]

\[
(1 - z)^{-a} =_{z} 1F_0(\{a\}; \{\}; z), \quad (B.3b)
\]

\[
\arcsin(z) =_{z} 2F_1\left(\left\{\frac{1}{2}, \frac{1}{2}\right\}; \left\{\frac{3}{2}\right\}; z^2\right). \quad (B.3c)
\]

If the same coefficient appears in the first and second list, the generalized hypergeometric function can be simplified to

\[
p_{q+1}F_{q+1}\left(\{a_1, \ldots, a_p, c\}; \{b_1, \ldots, b_q, c\}; z\right) =_{z} pF_q\left(\{a_1, \ldots, a_p\}; \{b_1, \ldots, b_q\}; z\right). \quad (B.4)
\]

The case \(p = q + 1\) is special. For \(\Re(\sum_{i=1}^{p} a_i - \sum_{i=1}^{q} b_i) > 0\), the generalized hypergeometric function is convergent for \(|z| \leq 1\). Especially for \(p = 1\), the value is

\[
2F_1(\{a, b\}; \{c\}; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \Re(c) > \Re(a + b). \quad (B.5)
\]

\(^1\)For a review, see [61–63].
B.2 Integrals

The above reviewed definitions and properties of hypergeometric functions can be used to calculate integrals. In this work, different integrals involving square roots appear. They can be rewritten using equation (B.3b) and sometimes simplified by reconstructing a hypergeometric series using equation (B.2).

B.2.1 Integral 1

The first type of integral we consider is

\[ I_1(a, s) = \int ds \frac{1}{\sqrt{1 - \frac{s^4}{z^4}}} \frac{1}{\sqrt{1 - s^6}} s^{a-1}, \quad a \neq 0. \]  

(B.6)

Using equation (B.3b), the integral can be written as a power series

\[ I_1(a, s) = \int ds \frac{1}{\sqrt{1 - \frac{s^4}{z^4}}} \ \binom{\frac{1}{2}}{\frac{1}{2}}_{(n)} \ \binom{\frac{1}{2}}{\frac{1}{2}}_{(m)} s^{4n+6m+a-1} z^{4n}, \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n! m!} \binom{\frac{1}{2}}{\frac{1}{2}}_{(n)} \ \binom{\frac{1}{2}}{\frac{1}{2}}_{(m)} \frac{1}{4n+6m+a} \ z^{4n+6m+a}. \]  

(B.7)

There are two values for \( s \) for which the result can be simplified. The first one is \( I_1(a, 0) \) or more precisely \( I_1(a, \epsilon) \) up to order \( \epsilon^0 \). The result is a finite number of terms. The second value is \( s = 1 \). I perform the first sum by using equation (B.2)

\[ I_1(a, 1) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n! m!} \binom{\frac{1}{2}}{\frac{1}{2}}_{(n)} \ \binom{\frac{1}{2}}{\frac{1}{2}}_{(m)} \frac{1}{4n+6m+a} \ z^{4n}, \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \binom{\frac{1}{2}}{\frac{1}{2}}_{(n)} \frac{1}{4n+a} \ z^{4n} \binom{2F_1}{\frac{1}{2}, \frac{1}{2}}_{\frac{2}{3}, \frac{2}{3}} \left( \binom{\frac{1}{2}, \frac{1}{2}}{\frac{2}{3}, \frac{2}{3}} ; \frac{1}{2} \right), \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \binom{\frac{1}{2}}{\frac{1}{2}}_{(n)} \frac{1}{4n+a} \ z^{4n} \sqrt{\pi} \Gamma (2n/3 + a/6 + 1), \]  

(B.8)

The same trick is used to perform the second sum. However, the redefinition \( n = \Delta n + 3\tilde{n} \) with \( \Delta n \in \{0, 1, 2\}, \ \tilde{n} \in \mathbb{N}_0 \) is necessary due to the Gamma functions. The
result simplifies to

\[ I_1(a, 1) = \sum_{\Delta n=0}^{2} \sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{(3n + \Delta n)!} \binom{1}{2} (3n+\Delta n) I \frac{1}{12\tilde{n} + 4\Delta n + a} \times \frac{\sqrt{\pi}\Gamma(2\Delta n/3 + a/6 + 1)}{\Gamma(2\tilde{n} + 2\Delta n/3 + a/6 + 1/2)}. \]

\[ = \sum_{\Delta n=0}^{2} \frac{\left(\frac{z_h}{\sqrt{\pi}}\right)^{4\Delta n}}{\Delta n!} \binom{1}{2} (\Delta n) \frac{1}{4\Delta n + a} \frac{\sqrt{\pi}\Gamma(2\Delta n/3 + a/6 + 1)}{\Gamma(2\tilde{n} + 2\Delta n/3 + a/6 + 1/2)} \times \times 6F5 \left( \left\{ \frac{1}{3}, \frac{1}{3} + \frac{\Delta n}{3}, \frac{2}{3} + \frac{\Delta n}{3}, 1 + \Delta n/3, \frac{a}{12} + 1/4 + \frac{\Delta n}{3}, \frac{a}{12} + 3/4 + \frac{\Delta n}{3} \right\}; \left(\frac{z_h}{\sqrt{\pi}}\right)^{12} \right). \]  

(B.9)

The result is the sum of three different generalized hypergeometric functions. For all values of \( \Delta n \) the generalized hypergeometric function simplifies and the result contains generalized hypergeometric functions up to \( _5F_4 \).

### B.2.2 Integral 2

A second type of integral is

\[ I_2(a, s) = \int ds \sqrt{1 - s^6} s^{a-1}. \]  

(B.10)

It can be solved in a similar fashion as \( I_1 \). For \( a \neq 0 \), the integral is

\[ I_2(a \neq 0, s) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{1}{2} \right)^{(n)} s^{6n+a-1} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{6n + a n!} \left( -\frac{1}{2} \right)^{(n)} s^{6n+a} , \]

\[ = \frac{s^a}{a} _2 F_1 \left( \left\{ -\frac{1}{2}, \frac{a}{6} \right\}, \left\{ \frac{a}{6} + 1 \right\}, s^6 \right). \]  

(B.11)

Again, this simplifies for \( s = \epsilon \) and \( s = 1 \)

\[ I_2(a \neq 0, \epsilon) = \frac{\epsilon^a}{a} + \mathcal{O}(\epsilon)^{a+6}, \]  

(B.12a)

\[ I_2(a \neq 0, 1) = \sqrt{\pi} \Gamma \left( \frac{a}{6} + 1 \right) \]  

(B.12b)

where equation (B.5) was used to obtain the value at unit argument.

For \( a = 0 \), the integral is to be solved carefully. The first term of the sum has to
be integrated separately and yields a logarithmic term. The result is
\[ I_2(0, s) = \int ds \sqrt{1 - s^6} s^{-1} \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{1}{2} \right)_n \int ds \ s^{6n-1}, \]
\[ = \ln(s) + s^6 \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \left( -\frac{1}{2} \right)_{m+1} \frac{1}{6(m+1)} s^{6m}, \]
\[ = \ln(s) - \frac{s^6}{12} F_2 \left( \left\{ \frac{1}{2}, 1, 1 \right\}, \left\{ 2, 2 \right\}, s^6 \right). \quad (B.13) \]

The values at \( s = 0 \) and \( s = 1 \) are
\[ I_2(0, \epsilon) = \ln(\epsilon) + O(\epsilon^1), \quad (B.14a) \]
\[ I_2(0, 1) = \frac{\ln(2) - 1}{3}. \quad (B.14b) \]

**B.2.3 Integral 3**

Another similar integral of this kind is
\[ I_3 = \int_{\epsilon/zh}^{1} ds \sqrt{1 - s^6} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n! m!} \left( -\frac{1}{2} \right)_m \left( \frac{1}{2} \right)_n \frac{1}{6m + 4n - 2} \left[ s^{6m+4n-2} \right]^{\frac{1}{2}}_{\epsilon/zh}, \]
\[ = \sum_{n=0}^{\infty} \frac{\sqrt{n}}{2n!} \frac{1}{4n - 2} \left( \frac{1}{2} \right)_n \frac{\Gamma \left( \frac{2(n+1)}{3} \right)}{\Gamma \left( \frac{2n}{3} + \frac{7}{6} \right)} + \frac{z^2}{2\epsilon^2}, \]
\[ = \frac{z^2}{2\epsilon^2} + \sum_{\Delta n=0}^{\infty} \left( \frac{1}{2} \right)_{\Delta n} \frac{\sqrt{n}}{4\Delta n - 2} \frac{\Gamma \left( \frac{2(\Delta n+1)}{3} \right)}{\Gamma \left( \frac{2\Delta n}{3} + \frac{7}{6} \right)} \frac{1}{2\Delta n!} \]
\[ \times 5 F_4 \left( \left\{ 1, \frac{2\Delta n - 1}{6}, \frac{2\Delta n + 1}{6}, \frac{2\Delta n + 3}{6}, \frac{2\Delta n + 5}{6} \right\}; \left\{ \frac{4\Delta n + 7}{12}, \frac{\Delta n + 2}{3}, \frac{\Delta n + 3}{3}, \frac{4\Delta n + 13}{12} \right\}; 1 \right), \]
\[ = -0.332947. \quad (B.15) \]

It can be solved in the same way as above. The first step is performing the summation over \( m \), then \( n \) is replaced by \( n = \Delta n + 3\tilde{n} \) and the last step is performing the summation over \( \tilde{n} \). There is no closed form for the remaining sum over \( \Delta n \), but the numerical value is known.
Appendix C

Shift of cut-off

When taking the backreaction into account, the field theory cut-off and the bulk cut-off are changed. In our calculation, the bulk cut-off is kept fixed. The remaining gauge freedom $z \rightarrow z + t_0 \xi(z)$ can be used to keep the field theory cut-off fixed at the same time. This determines the near boundary expansion of the metric perturbation.

First, it is important to understand the equivalence between the UV cut-off in the field theory and the IR cut-off in the bulk, which is explained in [64]. The holographic bound states that there should be no more than one degree of freedom per Planck area in the boundary field theory. The area is measured using the bulk metric. Since the metric is singular at the boundary, we introduce the bulk IR cut-off $\epsilon$.

\[
A = \text{Vol}(\mathbb{R}^3) \sqrt{|g_{xx}|}_{z=\epsilon}
\]  

(C.1)

On the field theory side, we introduce a UV cut-off at a distance scale $\delta$ to obtain a finite number of degrees of freedom. Introducing this cut-off can be understood as dividing the space into boxes of size $\delta^3$. In the large $N_c$ limit, each box contains $N_c^2$ degrees of freedom for the adjoint degrees of freedom. Hence, the number of degrees of freedom are

\[
N_{DOF} = N_c^2 \text{Vol}(\mathbb{R}^3) \frac{1}{\delta^3}.
\]  

(C.2)

This shows that the IR cut-off $\epsilon$ in the bulk corresponds to the UV cut-off at a distance scale $\delta$.

In [65], the exact correspondence between $\epsilon$ and $\delta$ was worked out using causality. The idea is to consider an excitation localized at the boundary $z = 0$ and to look at its propagation in the five-dimension AdS space. The construction is shown in Figure C.1. In the regularized bulk theory, the part of the AdS space with $z < \epsilon$ is ignored. Therefore, the excitation appears in the regularized theory at the time $t_1$ when the light-cone reaches the $z = \epsilon$ hypersurface. In the regularized field theory, all modes smaller than $\delta$ are ignored. Consequently, the excitation appears in the regularized theory at the time $t_2$ when the light-cone has the spatial extend $\delta$. Since both descriptions should be dual, we demand $t_1 = t_2$. 

Let us turn to our case. The backreacted metric is
\[
    ds^2 = \frac{L^2}{z^2} \left( \frac{1 + t_0 f(z)}{b(z)} dz^2 - (1 + t_0 h(z))b(z)dt^2 + (1 + t_0 j(z))d\vec{x}^2 \right),
\]
\[
b(z) = 1 - \frac{z^4}{z_h^4}.
\]
(C.3)

It is convenient to work in polar coordinates in the spatial directions.

\[
dx^2 = dr^2 + r^2 d\Omega_2^2
\]
(C.4)

In the following, we neglect higher order terms in $t_0$.

The first step is to write the spatial extent $\delta$ as an integral over time by using $ds^2 = 0$, $z = 0$ for the light-cone propagating on the boundary.

\[
\frac{1}{2} \delta^2 = \int_0^{\delta/2} dr,
\]
\[
= \int_0^{t_2} \left( 1 + \frac{t_0}{2} (h(0) - j(0)) \right) dt.
\]
(C.5)

The second step is to rewrite this as an integral over the radial coordinate $z$. For light propagating only in the $z$-direction (i.e. $ds^2 = 0$, $d\vec{x} = 0$), the substitution is

\[
dt = \frac{1}{b(z)} \left( 1 + \frac{t_0}{2} (f(z) - h(z)) \right) dz.
\]
(C.6)
The relationship between UV cut-off $\delta$ and IR cut-off $\epsilon$ is

$$
\frac{1}{2}\delta = \int_0^\epsilon \frac{1}{b(z)} \left( 1 + \frac{t_0}{2} (f(z) - h(z) + h(0) - j(0)) \right) dz. 
$$

(C.7)

The relevant part is the behaviour for small $\epsilon$

$$
\frac{1}{2}\delta = \int_0^\epsilon dz \left[ 1 + \frac{t_0}{2} \left( f(0) - j(0) + (f'(0) - h'(0)) z + \frac{f''(0) - h''(0)}{2} z^2 \right) + \mathcal{O}(z)^3 \right],
$$

$$
\left. \frac{1}{2}\delta \right|_{t_0=0} + \frac{t_0}{2} \epsilon (f(0) - j(0)) + \frac{t_0}{4} \epsilon^2 (f'(0) - h'(0)) + \frac{t_0}{12} \epsilon^3 (f''(0) - h''(0)) + \mathcal{O}(\epsilon)^4,
$$

(C.8)

The field theory cut-off $\delta$ should agree in the perturbed and in the unperturbed geometry. This matching determines the boundary expansion of $j$. The boundary values in equations (4.21) and (4.22) yield

$$
0 = \frac{t_0}{2} \epsilon \left( \frac{1}{12} - j(0) \right) - \frac{t_0}{2} \epsilon^2 j'(0) - \frac{t_0}{12} \epsilon^3 \left( \frac{\theta'(0)^2}{3} + 3 j''(0) \right) + \mathcal{O}(\epsilon)^4
$$

$$
j(\epsilon) = \frac{1}{12} - \frac{\theta'(0)^2}{18} \epsilon^2 + \mathcal{O}(\epsilon)^4.
$$

(C.9)
C. Shift of cut-off
Appendix D

Numerical calculations

The numerical calculations in this thesis were done in Mathematica 10.3. In the following, I summarize the important parts of my program. It splits in two parts: the calculation of the backreaction $\kappa_1$ and $\kappa_2$ and the flavour correction to the entanglement entropy $\delta S_{EE}$.

To be able to change the accuracy and precision globally, they are defined in the preamble. Also, sometimes we need approximately zero or $\infty$ to avoid singularities during the numerical calculations. For that, we define zero and infty.

$\text{WorkingPrecision} = 30; \text{PrecisionGoal} = 20; \text{AccuracyGoal} = 20;$

\(\text{infty} = 10^6; \text{one} = 1; \text{zero} = 0;\)

Furthermore, the differential equations are solved for $zh = 1$, i.e. $T = 1/\pi$. The temperature dependence is restored later.

\(b[z_] = 1 - z^4; zh = 1;\)

D.1 Calculation of backreaction

When calculating the backreaction on the metric, the first step is to re-derive the embedding of the probe branes. For the initial conditions, the end of the branes $z_0$ for Minowski embeddings and the value $\theta_0 = \theta(z_h)$ for the black hole embeddings has to be fixed (c.f. initial conditions in equations (2.126) and (2.127)). I solved the equation of motion $\text{eom}[\theta[z]] = 0$ numerically for the appropriate initial conditions.

\(\text{SolutionMinkowski}[n_?\text{NumberQ}, z_]:=\theta[z]/.\text{First}[\text{NDSolve}[\{\text{eom}[\theta[z]] == 0, \\
\theta[z_0[[n]]] == (\text{Pi}/2 - \text{zero}), \theta'[z_0[[n]]] == \text{infty}] , \theta[z], \{z, \text{zero}, z_0[[n]]\}, \ldots ]]\)

\(\text{SolutionBH}[n_?\text{NumberQ}, z_]:=\theta[z]/.\text{First}[\text{NDSolve}[\{\text{eom}[\theta[z]] == 0, \\
\theta[\text{one}] == \theta_0[[n]], \theta'[\text{one}] == 3\text{Tan}[\theta_0[[n]]]/4}, \theta[z], \{z, \text{zero, one}\}, \ldots ]]\)
The previous defined function where then applied to the fixed initial conditions. The list of embeddings is saved as association to keep track of $z_0$ and $\theta_0$. Near the phase transition, the mass can not be used as a unique parameter, but this variables can.

\[
\text{MinkowskiEmbeddings} = \text{Association}[
\text{ParallelTable}[\kappa, \{n\}]
\]

\[
\text{Evaluate}[\text{Solution}\theta\text{Minkowski}[n, z]], \{n, 1, \text{Dimensions}[z_0[[1]]]\}];
\]

\[
\text{BHEmbeddings} = \text{Association}[
\text{ParallelTable}[\theta_0[[n]] \rightarrow \text{Function}[\{z\}], \{n, 1, \text{Dimensions}[\theta_0[[1]]]\}];
\]

For validity, I required that $\theta(0) < \text{zero}$ and $\theta''(0) < \text{zero}$. In the end, I had 89 Minkowski embeddings and 73 black hole embeddings to analyse the general behaviour and 86 and 77 embeddings near the critical embedding.

This solutions to the embedding were then used to calculate the backreaction. The differential equations for the $\kappa_1$ and $\kappa_2$ ($\kappa1DGL = 0$ and $\kappa2DGL = 0$ respectively) were solved numerically for the appropriate initial conditions. For the black hole embedding, this can be solved on the whole interval.

\[
\text{Solution}\kappa1\text{Minkowski}[z_0] :=
\text{NDSolve}[[0 == \kappa1DGL[z]/.\theta \rightarrow \text{tmp} , \kappa1[z_0] == 0], \kappa1[z], \{z, 0, z_0\}, \ldots][[1]]
\]

\[
\text{Minkowski}\kappa1 = \text{Association}[
\text{ParallelTable}[n \rightarrow \text{Function}[\{z\}], \text{Evaluate}[\kappa1[z]];\]

\[
\text{Solution}\kappa2\text{Minkowski}[n, \text{MinkowskiEmbeddings}[n]]], \{n, \text{Keys[. . .]}}];
\]

\[
\text{Solution}\kappa2\text{BH}[\theta_0] := \text{NDSolve}[
\{0 == \kappa2DGL[z]/.\theta \rightarrow \text{tmp} , \kappa2[z_0] == 0, \kappa2[z_0] == 0\}, \kappa2[z], \{z, 0, z_0\}, \ldots][[1]]
\]

\[
\text{BH}\kappa2 = \text{Association}[
\text{ParallelTable}[n \rightarrow \text{Function}[\{z\}], \text{Evaluate}[\kappa2[z]];\]

\[
\text{Solution}\kappa2\text{BH}[n, \text{BHEmbeddings}[n]]], \{n, \text{Keys[BHEmbeddings]}}];
\]
D.2 Calculation of entanglement entropy

Since for $\kappa_2$ only the boundary value is important for the entanglement entropy, it is saved in $\kappa_{20\text{Minkowski}}$ and $\kappa_{20\text{BH}}$.

D.2 Calculation of entanglement entropy

The cut-off independent part of the correction to the entanglement entropy is

$$\delta_1 S_{EE}(B) = \frac{t_0L^3\ell^2}{96G_N} \left( -\frac{4}{3}m^2 - \frac{1}{\min(z_0, z_*)^2} + 4m^2 \log \left( \frac{z_h}{\min(z_0, z_*)} \right) \right)$$

$$+ \int_0^{\min(z_0, z_*)} dz \frac{t_0L^3\ell^2}{4G_N} \left( \frac{1}{z^3b(z)^{3/2}}\kappa_1(z) - \frac{\kappa_1(0)}{z^3} - \frac{\kappa_1'(0)}{z^2} - \frac{\kappa_1''(0)}{2z} \right).$$

(D.1)

The result is proportional to $\text{ConstantsdS} = \lambda N_c N_f \ell^2$. Furthermore, due to the choice $z_h = 1$, one has to add the factor $1/z_h^2$ to obtain the general temperature result. This introduces the additional factors of $\pi^2$. Therefore, the numerical results are in units of $\lambda N_c N_f \ell^2 T^2$. $\delta_2 S_{EE} = \delta 2S$ is calculated analytically as function of $\kappa_2(0)$. It is also possible to use $\delta_2 S_{EE}$ to also obtain the volume term, but this result is not trustworthy in the limit $z_* \to z_0$. Therefore, to obtain the asymptotic value it is important to take the flavour correction to the ‘corrected’ entanglement entropy. In both cases, the flavour correction $\delta S_{EE}$ is calculated point-wise through the functions


Re[Simplify[BoundaryTerms[z_, key_]/ConstantsdS /. κ1 -> Minkowskiκ1[key]]

+ NIntegrate[Simplify[Integrand[x, z*]/ConstantsdS /. κ1 -> Minkowskiκ1[key]]

, {x, 0, Min[key, z*]}, ...] + Simplify[1/ConstantsdS δ2S[z*, κ_{20\text{Minkowski}}[key]]]]; and


Re[Simplify[1/ConstantsdSBHboundaryTerm[z*, 1] /. κ1 -> BHκ1[key]] + NIntegrate[

Simplify[Integrand[x, z*]/ConstantsdS /. κ1 -> BHκ1[key]], {x, zero, z*}, ...]

+ Simplify[1/ConstantsdS δ2S[z*, κ_{20\text{BH}}[key]]];

Later, this is interpolated by the following functions

MinkowskiDeltaS = Association[ParallelTable[n → FunctionInterpolation[...]];
Evaluate[SolutionDeltaSMinkowski[z\*, n], \{z\*, zero, one\}, \{n, Keys[Minkowski\[\kappa\]]\}];

BHDeltaS = Association[ParallelTable[n -> FunctionInterpolation[Evaluate[SolutionDeltaSBH[z\*, n], \{z\*, zero, one\}], \{n, Keys[BH\[\kappa\]]\}];]
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Ehrenwörtliche Versicherung

Ich versichere hiermit, dass diese Masterarbeit von mir selbstständig angefertigt und alle verwendeten Quellen und Hilfsmittel angegeben wurden.

Desweiteren versichere ich, dass diese Arbeit bisher keiner weiteren Prüfungsbehörde vorgelegt und auch nicht veröffentlicht wurde.

München, den ______________

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Nina Miekley